Prop: If $\sum_{n=0}^{\infty} a_n x_0^n$ converges for some $x_0 \neq 0$ and if $0 < R < |x_0|$ then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-R, R]$.

Pf: Because $\sum_{n=0}^{\infty} a_n x_0^n$ converges there exists $C >$ with

$|a_n x_0^n| \leq C$ for all n. $\left(a_n x_0^n \to 0 \right)$.

Pick R with $0 < R < x_0$ and let $r = \frac{R}{|x_0|}$ so $R = r|x_0|$

and $0 < r < 1$. If $x \in [-R, R]$ then

$$
\begin{aligned}
|a_n x^n| &= |a_n| \, |x|^n \\
&\leq |a_n| \, R^n \\
&= |a_n| \, |x_0|^n r^n
\end{aligned}
$$

$$= |a_n x_0^n| \, r^n$$

$$\leq C r^n.$$

$f_n(x) = a_n x^n$

$|f_n(x)| \leq \underbrace{C r^n}_{M_n}$

$\forall x \in [-R, R]$

Because $0 < r < 1$, $\sum_{n=0}^{\infty} C r^n$ converges.

Hence, by the Weierstrass M-test ($M_n = C r^n$)

the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R, R]$. $\square$

1) Each f_n is cts. and differentiable on $[a,b]$ ✓

2) Each f_n' is continuous on $[a,b]$ (*) ✓

3) $f_n' \to g$ uniformly for some g ✓

4) $f_n(x_0) \to c$ for some $c \in \mathbb{R}$ and $x_0 \in [a,b]$ ✓✓✓

$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{Converges at } x = x_0 \neq 0 \qquad 0 < R < |x_0|$$

$$f_n(x) = \sum_{k=0}^{n} a_k x^k \qquad f_n \to f \text{ uniformly on } [-R,R]$$

$$f_n'(x) = \sum_{k=1}^{n} k\, a_k\, x^{k-1}$$

f_n' is the partial sum of the series $\sum_{k=1}^{\infty} k\, a_k\, x^{k-1}$

We need to show that $\sum_{k=1}^{\infty} k a_k x^{k-1}$ converges uniformly

to some g.

We're looking for constants M_k such that

$$|k a_k x^{k-1}| \leq M_k \quad \text{for all } x \in [-R,R]$$

and such that $\sum_{k=1}^{\infty} M_k$ converges.

$$k(k-1) a_k x^{k-2}$$

$$\begin{aligned} |k\, a_k x^{k-1}| &= k\, |a_k x^{k-1}| \\ &\leq k\, |a_k|\, R^{k-1} \end{aligned}$$

$$= k\,|a_k|\,r^{k-1}\,|x_0|^{k-1}$$

$$= k\,\frac{|a_k|\,|x_0|^k}{|x_0|}\,r^{k-1}$$

$$\leq k\,\frac{C}{|x_0|}\,r^{k-1}$$

$\sum M_k$ converges

Let $M_k = k\,\frac{C}{|x_0|}\,r^{k-1}$.

$$\sum_{k=1}^{\infty} k\,r^{k-1}$$

$$\sum_{n=1}^{\infty} b_n \qquad \underbrace{\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|}_{r} < 1 \Rightarrow \text{series converges.}$$

$b_{n+1} \sim r\,b_n$ for large n

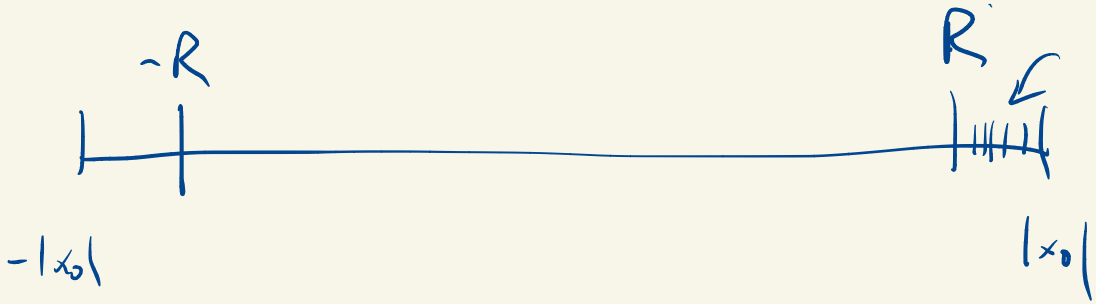
$$\lim_{k\to\infty} \frac{(k+1)\, r^{k}}{k\, r^{k-1}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right) r = r < 1.$$

So: $\sum_{k=1}^{\infty} M_k$ converges $\left(M_k = k \frac{C}{|x_0|} r^{k-1} \right)$

By the WMT, $\sum_{k=1}^{\infty} k a_k x^{k-1}$ converges uniformly on $[-R, R]$ to some limit g.

$\Rightarrow$ f is differentiable on $[-R,R]$ and

$$f'(x) = \sum_{k=1}^{\infty} k\, a_k x^{k-1}.$$



f

$$R_1 > R_2 > \dots > R_q = R$$

$C[0,1]$, the metric space (L_∞ norm)
(normed linear space)

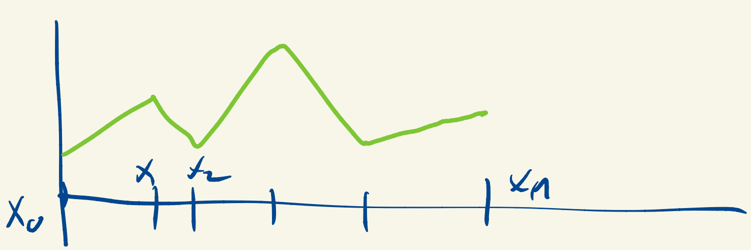
- dense, useful subsets
- how to identify compact subsets

$P[0,1]$ polynomials on $[0,1]$

$PL[0,1]$ piecewise linear continuous functions on $[0,1]$

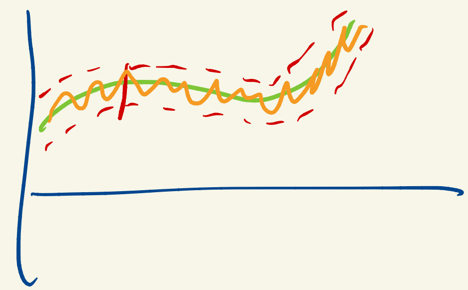
$f \in PL[0,1]$ if there exist $0 = x_0 < x_1 < \cdots < x_n = 1$

and $f\big|_{[x_{k-1}, x_k]}$ is linear for each $1 \leq k \leq n$,



Exercise $P[0,1]$ and $PL[0,1] \subseteq C[0,1]$

and indeed are subspaces.



Claim 1) $\overline{P[0,1]} = C[0,1]$

2) $\overline{PL[0,1]} = C[0,1]$

We'll prove 2) first.

We'll also show $\overline{P[0,1]} \supseteq PL[0,1]$

$$\Rightarrow C[0,1] \supseteq \overline{P[0,1]} = \overline{\overline{P[0,1]}}$$

$$\supseteq \overline{PL[0,1]}$$

$$= C[0,1]$$

This step will be a little abstract.

If W is a subspace V then $\overline{W}$ is also a subspace.

Example of a normed vector space V and a ~~subset~~ subspace W that is not closed in V.

$Z \to$ sequences ending in all 0's.

$Z \subseteq \ell_1$, Z is a subspace

$\overline{Z} = \ell_1$

Let $x = (x_1, x_2, \ldots) \in \ell_1$

Let $\varepsilon > 0$. Find N so that $\sum_{n=N}^{\infty} |x_n| < \varepsilon$.

Let $z = (x_1, \ldots, x_{N-1}, 0, 0, \ldots)$ so $z \in Z$, and

$\|z - x\|_{\ell_1} < \varepsilon$.