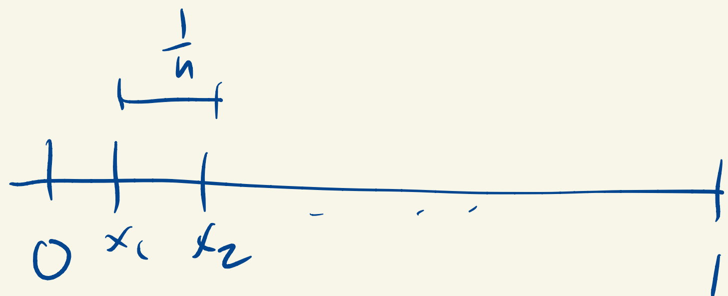
Prop: $\overline{PL[0,1]} = C[0,1]$

Pf: Let $f \in C[0,1]$ and let $\varepsilon > 0$.

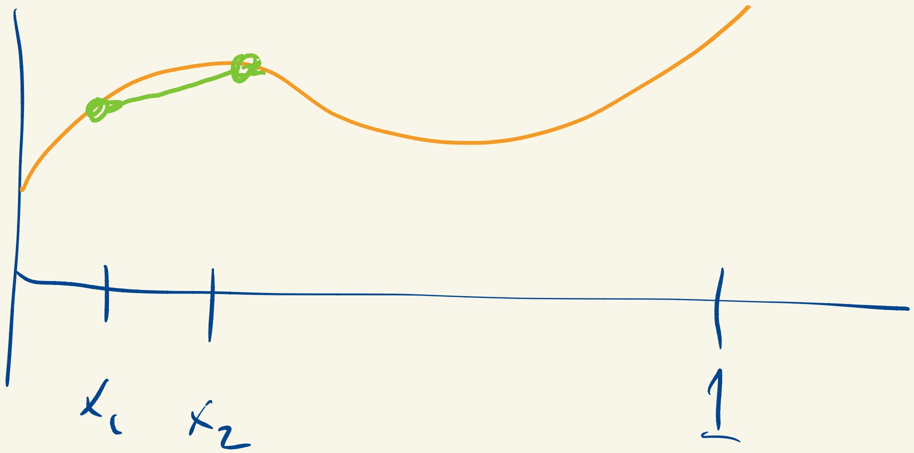
Job: Find $g \in PL[0,1]$ s.t. $\|g - f\|_\infty < \varepsilon$.

Since $[0,1]$ is compact, f is uniformly continuous so we can find $n \in \mathbb{N}$ such that if $a, b \in [0,1]$ and $|b-a| \leq \frac{1}{n}$ then $|f(b) - f(a)| < \varepsilon$.



Let $x_k = \frac{k}{n}$ $\quad 0 \leq k \leq n$.

Let g be the piecewise linear function that equals f at each x_k and that is linear on each $[x_{k-1}, x_k]$.



Observe that on $[x_{k-1}, x_k]$

$$g(x) = f(x_{k-1}) + \frac{x - x_{k-1}}{x_k - x_{k-1}}\left(f(x_k) - f(x_{k-1})\right)$$

Let $x \in [0,1]$ and pick k so that $x \in [x_{k-1}, x_k]$.

Then
$$|f(x) - g(x)| = \left| f(x) - \left[f(x_{k-1}) + \frac{x - x_{k-1}}{x_k - x_{k-1}}\left(f(x_k) - f(x_{k-1})\right)\right]\right|$$

$$\leq |f(x) - f(x_{k-1})| + \left|\frac{x - x_{k-1}}{x_k - x_{k-1}}\right| |f(x_k) - f(x_{k-1})|$$

$$\leq \varepsilon + 1\,\varepsilon$$

$= 2\varepsilon.$

Hence $\|f-g\|_\infty \le 2\varepsilon.$ $\square$

We showed that PL functions with partitions at rational endpoints are dense.

Exercise: show that the PL functions as with rational values at the partition points are dense.

Consequence: $C[0,1]$ is separable. $\left(c_0 \subseteq \ell_\infty \right)$

Exercise: Show that $B[0,1]$ (bounded functions on $[0,1]$) is not separable.

Lemma: Let X be a normed vector space and W a subspace.

Then $\overline{W}$ is a subspace.

Pf: Let $a, b \in \overline{W}$. There exist sequences (a_n) and (b_n)

in W converging to a, b. Then $(a_n + b_n)$ is a

sequence in W and $a_n + b_n \to a + b$.

Hence $a+b \in \overline{W}$.

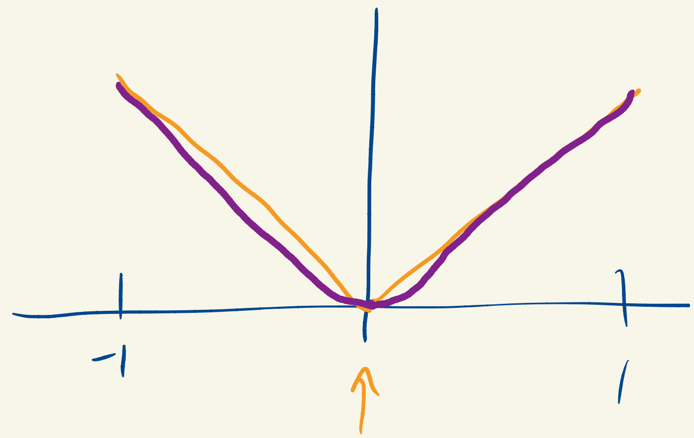
Scalar multiplication is handled similarly.

Hard fact:

Prop: Given $\varepsilon > 0$ there exists a polynomial p such that

$$\left| \operatorname{abs}(x) - p(x) \right| < \varepsilon \text{ for all } x \in [-1, 1].$$

$$\left(\operatorname{abs} \in \overline{P[-1,1]} \right)$$

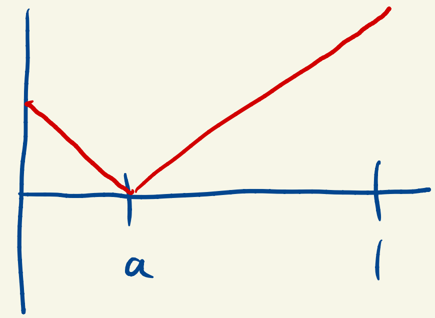


We'll show that

$$PL[0,1] \subseteq \overline{P[0,1]}$$

$\text{abs}_a(x) = \text{abs}(x-a)$ $[0,1]$

$0 \le a \le 1$



$\text{abs}_a \in \overline{P[0,1]}$

Let $\varepsilon > 0$. Find $q \in P[-1,1]$ with $\| \text{abs} - q \|_\infty < \varepsilon$.

on $[-1,1]$

Then if $x \in [0,1]$ then $x - a \in [-1,1]$

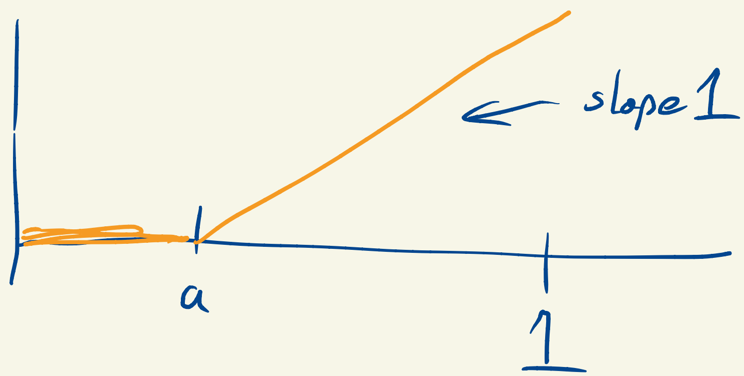
So $| \text{abs}(x-a) - q(x-a) | < \varepsilon$

i.e. $| \text{abs}_a(x) - p(x) | < \varepsilon$ for all $x \in [0,1]$.

$\| \mathrm{abs}_a - p \|_\infty < \varepsilon.$

R_a

"ramp"



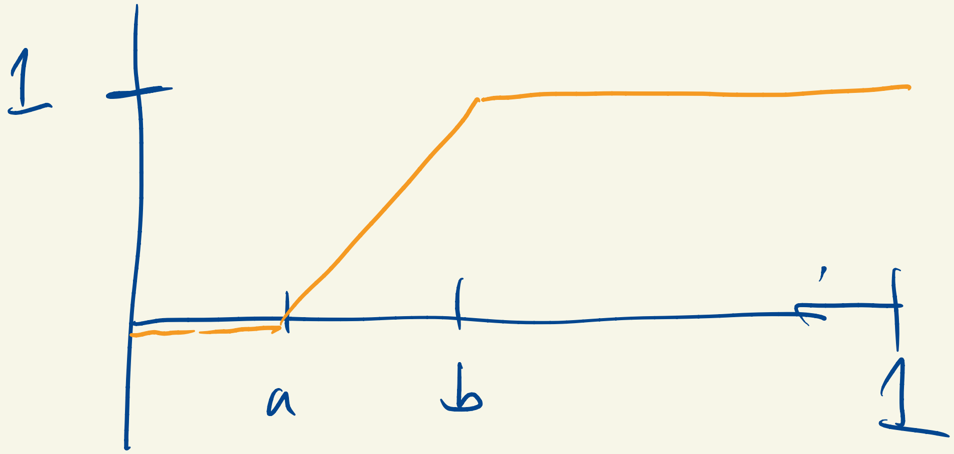
$0 \le a \le 1$

$R_a \in \overline{P[0,1]}$

linear combo of things in $\overline{P[0,1]}$

$$R_a(x) = \frac{1}{2}\left((x-a) + \mathrm{abs}_a(x) \right)$$

$$\mathrm{abs}_a(x) = \begin{cases} x-a & x \ge a \\ -(x-a) & x < a \end{cases}$$



$0 \le a < b \le 1$

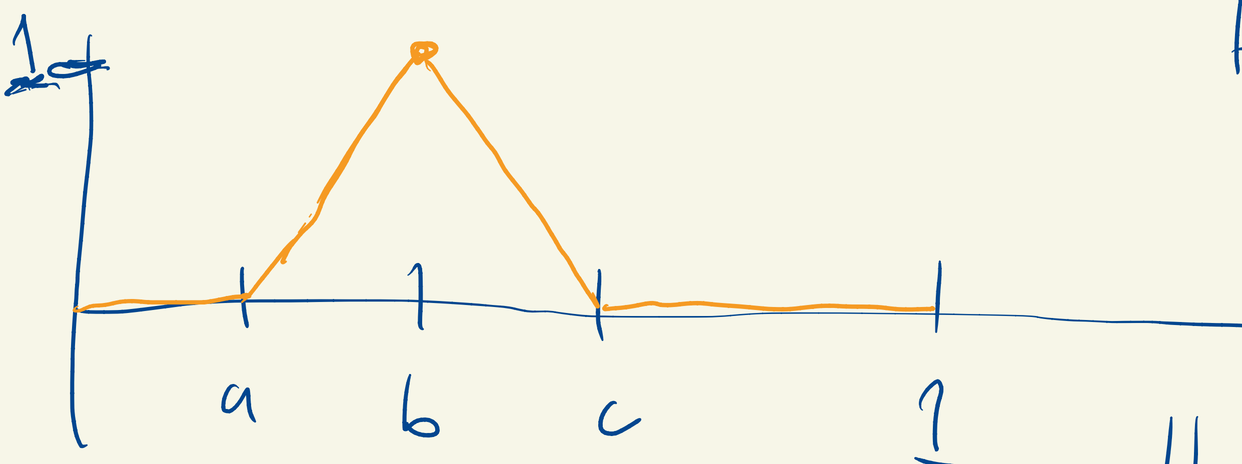
$$V_{a,b} = \frac{R_a - R_b}{b - a}$$

$$V_{a,b} \in \overline{P[0,1]}$$

"hat"

$a < b < c$

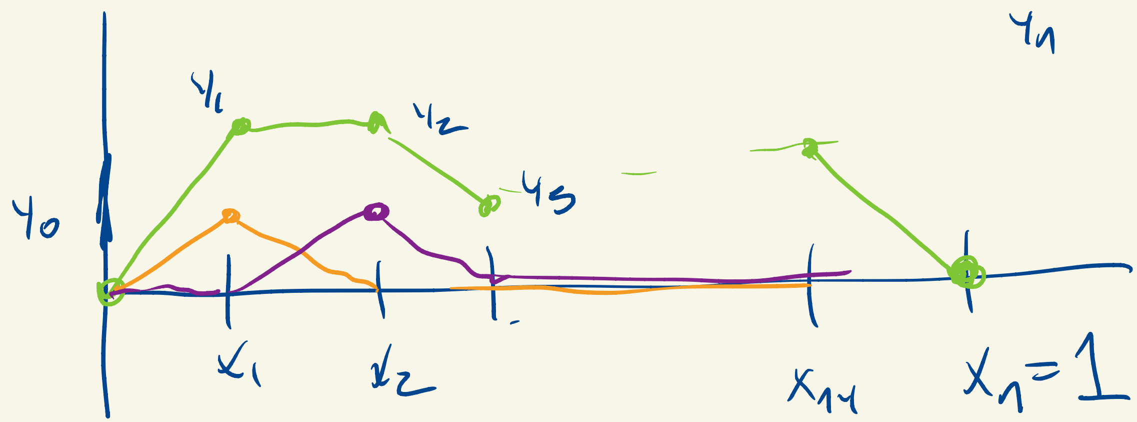
$H_{a,b,c}$



$$H_{a,b,c} = V_{a,b} - V_{b,c}$$

$H_{a,b,c} \in \overline{P[0,1]}$ since $V_{a,b}$ and $V_{b,c}$ are,

$f \in PL[0,1]$, $f(0)=0$, $f(1)=0$



$$\sum_{k=1}^{n-1} y_k H_{x_{k-1}, x_k, x_{k+1}}(x) = f(x)$$

linear combo of hat functions, so is in $\overline{P[0,1]}$

Any $f \in PL[0,1]$ is a linear combination of a linear function and a PL function that vanishes at 0 and 1.

Prop: Given $\varepsilon > 0$ there exists a polynomial $p(x)$ with

$$|\operatorname{abs}(x) - p(x)| < \varepsilon \quad \text{for all } x \in [-1,1].$$

Pf: Let $P_0(x) = 0$ and if P_n has been defined, let

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n(x)^2\right)$$

Claim $P_n \to$ sqrt uniformly on $[0,1]$.

We first show that for all n,

$$P_{n+1} \geqslant P_n \geqslant 0$$

and that $\sqrt{x} \geqslant P_n(x) \geqslant 0$ for all $x \in [0,1]$.

These facts are clear if $n=0$.

Suppose these facts hold for some n.

Observe that

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(\sqrt{x} + P_n(x)\right) \cdot \left(\sqrt{x} - P_n(x)\right).$$

Moreover $\frac{1}{2}\left(\sqrt{x} + P_n(x)\right) \leq \frac{1}{2}\left(\sqrt{x} + \sqrt{x}\right) \leq 1$

and $\sqrt{x} - P_n(x) \geqslant 0$.

Hence $P_{n+1}(x) \leq P_n(x) + 1 \cdot \left(\sqrt{x} - P_n(x)\right)$

$$= \sqrt{x}.$$

Moreover, $P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n(x)^2\right)$

$$\geq P_n(x) \geq 0.$$

Finally, $P_{n+2}(x) \geq P_{n+1}(x)$ for the same reasons.

Now for each $x \in [0,1]$, $P_n(x)$ is a monotone increasing sequence in $\mathbb{R}$ that is bounded above by $\sqrt{x}$. So it converges to some $f(x)$.

That is $P_n \to f$ pointwise on $[0,1]$.

Using

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n(x)^2\right)$$

we take a limit and conclude

$$f(x) = f(x) + \frac{1}{2}\left(x - f(x)^2\right)$$

so $f(x)^2 = x$. But $f(x) \geq 0$, so $f(x) = \sqrt{x}$.

Since $\sqrt{x}$ is continuous and since P_n is monotone increasing in n, $P_n \to \sqrt{a}$ uniformly.