Using

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}\left(x - P_n(x)^2\right)$$

we take a limit and conclude

$$f(x) = f(x) + \frac{1}{2}\left(x - f(x)^2\right)$$

so $\quad f(x)^2 = x$. But $f(x) \geq 0$, so $f(x) = \sqrt{x}$.

Since $\sqrt{x}$ is continuous and since P_n is monotone increasing in n, $P_n \to \sqrt{x}$ uniformly. (Dini's Theorem)

and since $[0,1]$ is compact

$P_n \nearrow f$ $\qquad f(x) = \sqrt{x}$

pointwise on $[0,1]$.

$X = [0,1]$, cpct. ✓

Dini's thm: (f_n) $\quad f_n : X \to \mathbb{R}$ $\quad \underbrace{f_n \in C(X)}_{\text{polynomials}}$ ✓

↳ cpct

$\underbrace{f_n(x) \text{ is monotone in } n}$ → ✓

$\underbrace{f_n \to f \text{ pointwise}}$, and $\underbrace{f \in C(X)}$ ✓

$\Rightarrow f_n \to f$ uniformly.

Now let $\varepsilon > 0$ and find a polynomial p such that

$$\left|\sqrt{x} - p(x)\right| < \varepsilon \quad \text{for all } x \in [0,1].$$

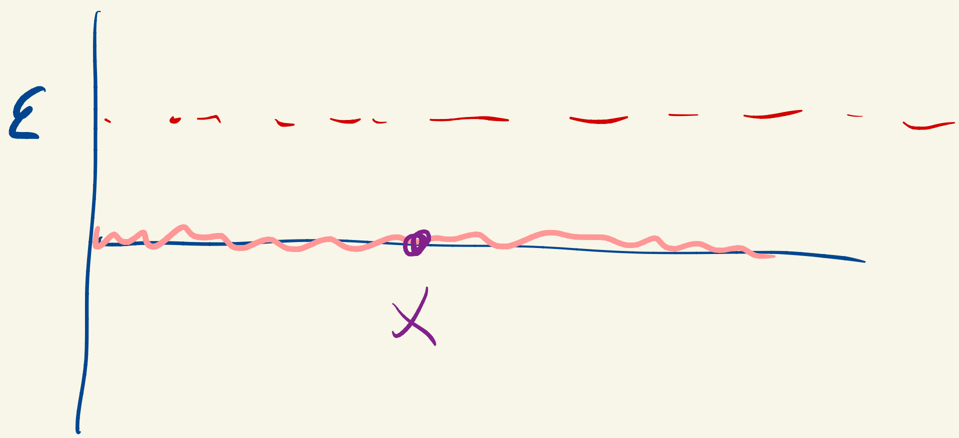
But then $\left|\sqrt{x^2} - p(x^2)\right| < \varepsilon$ for all $x \in [-1,1]$.

Since $\sqrt{x^2} = |x|$ the proof is complete. $\square$

U_n: good places for f_n

$x \in U_n$ for some n?

$f_n(x) < \varepsilon$



$$\int_0^{1-1/n} f_n \qquad \int_0^1 f_n = \underline{\underline{\int_0^{1-1/n} f_n}} + \int_{1-1/n}^1 f_n$$

$$\left| \int_0^{1-1/n} f_n - \int_0^1 f \right|$$

$$\left| \int_0^1 f_n \overbrace{\boxed{- \int_{1-1/n}^1 f_n}}^{B} - \int_0^1 f \right|$$

$$|A + B - A'| \leq |A - A'| + |B|$$

Trig polynomials

$$T(x) = a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

Certainly these are continuous and 2π-periodic

$$T(x+2\pi) = T(x) \quad \forall x \in \mathbb{R}.$$

$C^{2\pi}$

Claim: $f \in C^{2\pi} \Rightarrow \underline{f}$ is bounded. $(f \in B(\mathbb{R}))$

$$\left\| f|_{[-\pi,\pi]} \right\|_\infty = \|f\|_\infty$$

We'll put the $\|\cdot\|_\infty$ on $C^{2\pi}$.

We can identify $C^{2\pi}$ with a closed subspace of $C[-\pi,\pi]$.

$$f \in C^{2\pi} \longrightarrow f\big|_{[-\pi,\pi]}$$

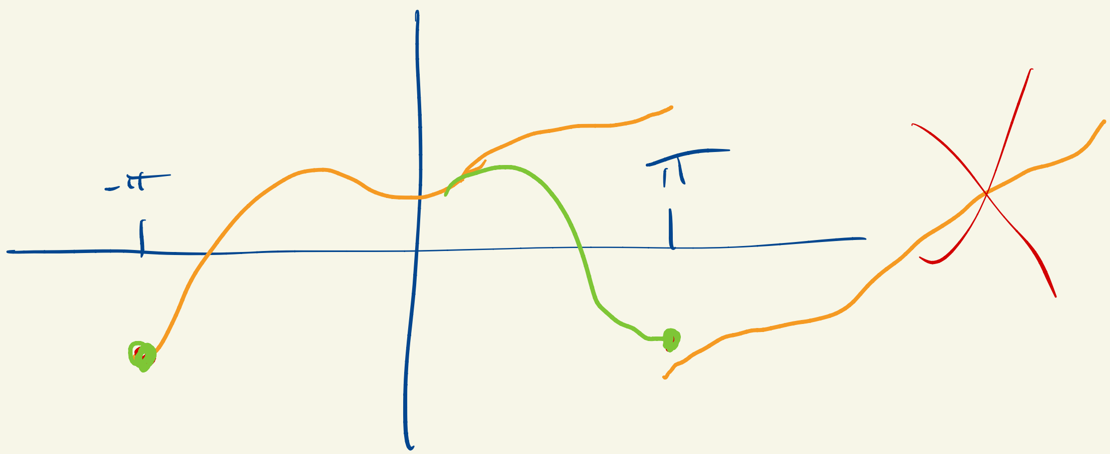


Image is the set of functions with $f(-\pi) = f(\pi)$.

Exercise: This is indeed a closed subspace of $C[-\pi,\pi]$.

Goal: $\mathcal{T} = \{\text{trig polynomials}\} \subseteq C^{2\pi}$ (and is a subspace)

$$\overline{\mathcal{T}} = C^{2\pi}$$

Key observations

1) $\mathcal{T}$ is not just a subspace. It's closed under multiplication. (I.e. it's a algebra!)

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B)$$

$$\leadsto e^{iA}e^{iB} = (\cos(A) + i\sin(A))(\longrightarrow)$$

$$\sin(kx)\sin(mx) = \frac{1}{2}\left[\cos((k-m)x) - \cos((k+m)x)\right]$$

with similar formulas for $\sin(kx)\cos(mx)$

$\cos(kx)\cos(mx)$

2) $\mathcal{T}$ is closed under translation by $\pm\frac{\pi}{2}$

$$\sin\left(k\left(x+\frac{\pi}{2}\right)\right) = \sin\left(kx + \frac{k\pi}{2}\right)$$

$$= \begin{cases} \sin(kx) & k = 0 \mod 4 \\ \cos(kx) & k = 1 \mod 4 \\ -\sin(kx) & k = 2 \mod 4 \\ -\cos(kx) & k = 3 \mod 4 \end{cases}$$

and similarly for $\cos(kx)$.

Lemma: Suppose $f \in C^{2\pi}$ and is even (so $f(-x) = f(x)\ \forall x \in \mathbb{R}$).
Then $f \in \overline{\mathcal{T}}$.

Supposing I prove the lemma, let's show $\overline{\mathcal{T}} = C^{2\pi}$

Start with $f \in C^{2\pi}$

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

$$= \underbrace{f_e(x)}_{\text{even}} + \underbrace{f_o(x)}_{\text{odd}}$$

$f_e \in \overline{\mathcal{T}}$. Also $f_o(x) \cdot \sin(x)$ is even.

So $f_o \cdot \sin \in \overline{\mathcal{T}}$.

$f_e \cdot \sin^2 \in \overline{\mathcal{T}}$] $\qquad f_o \cdot \sin^2 \in \overline{\mathcal{T}}$

Exercise: (If $g \in \overline{\mathcal{T}}$ then $g \cdot T \in \overline{\mathcal{T}}$ for any $T \in \overline{\mathcal{T}}$)

$$g \cdot T \qquad T' \cdot T$$

$$f \cdot \sin^2 = (f_e + f_o)\sin^2 \in \overline{T}.$$

$$\tilde{f}(x) = f\left(x - \tfrac{\pi}{2}\right)$$

$$\tilde{f} \cdot \sin^2 \in \overline{T}$$

$$\tilde{f}\left(\underline{x + \tfrac{\pi}{2}}\right) \cdot \sin^2\left(x + \tfrac{\pi}{2}\right) \in \overline{T}$$ (uses the earlier observation about translation)

$$f \cdot \cos^2 \in \overline{T}$$

$$f \cdot \cos^2 + f \cdot \sin^2 \in \overline{T} \Rightarrow f \in \overline{T}.$$

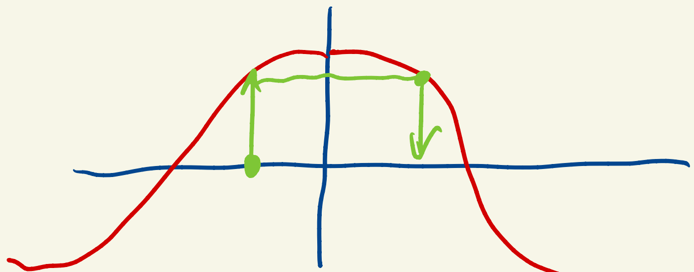
Lemma: Suppose $f \in C^{2\pi}$ and is even (so $f(-x) = f(x)\ \forall x \in \mathbb{R}$).
Then $f \in \overline{\mathcal{T}}$.

Pf: Consider $f \circ \arccos : [-1,1] \to \mathbb{R}$. Let $\varepsilon > 0$.

Find a polynomial p such that

$$|f(\arccos(y)) - p(y)| < \varepsilon \quad \text{for all } y \in [-1,1].$$

Now observe $\arccos(\cos(x)) = \begin{cases} x & 0 \le x \le \pi \\ -x & -\pi \le x \le 0. \end{cases}$



I.e. $\arccos(\cos(x)) = |x|$
for $x \in [-\pi, \pi]$.

Hence $\quad \Big| f(|x|) - \underbrace{p(\cos(x))}_{T(x)} \Big| < \varepsilon$ for all $x \in [-\pi, \pi]$.

Since f is even, $f(|x|) = f(x)$ for all $x \in [-\pi, \pi]$.

So $\quad | f(x) - T(x) | < \varepsilon$ for all $x \in \underline{[-\pi, \pi]}$

and we are done. $\square$

$$\| f - T \|_{C^{2\pi}} = \| f - T \|_{C[-\pi, \pi]}$$

$$\mathrm{Re}\left(p(e^{ix}) \right)$$

Thm (Weierstrass)

Let $f \in C^{2\pi}$. Given $\varepsilon > 0$ there is a trig polynomial T with $\|f - T\|_\infty < \varepsilon$.

$$f \in \overline{\mathcal{T}} \qquad T \in \mathcal{T} \implies f - T \in \overline{\mathcal{T}}$$

$$\|T\|_\infty \leq M$$

$$T' \qquad \|f - T'\|_\infty < \frac{\varepsilon}{M}$$

$$\|(f - T')T\|_\infty < \varepsilon$$

$$\|fT - \underbrace{T'T}_{\in \mathcal{T}}\|_\infty < \varepsilon$$

$C(X)$ $\qquad$ A algebra $\subseteq C(X)$

contains constants

separates points

Given $x_1, x_2 \in X$ $\qquad x_1 \neq x_2$

there exists $f \in A$ with $f(x_1) \neq f(x_2)$

$$\overline{A} = C(\underbrace{X}_{\text{compact}})$$

$[0,1]$

$[a,b]$

Stone-Weierstrass

$C[-\pi, \pi]$