$C(X)$

A algebra $\subseteq C(X)$

contains constants

separates points: Given $x_1, x_2 \in X$, $x_1 \neq x_2$

there exists $f \in A$ with $f(x_1) \neq f(x_2)$

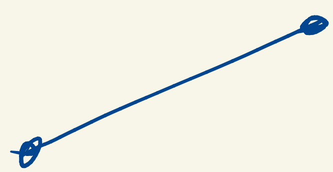
$$\overline{A} = C(\underbrace{X}_{\text{compact}})$$

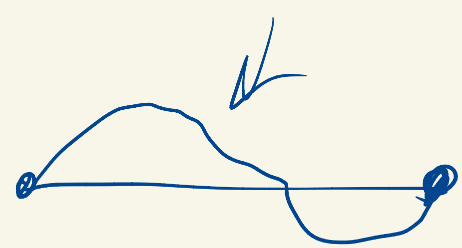
$[0,1]$

$[a,b]$

Stone-Weierstrass

$C[-\pi, \pi]$





$C[0,1]$ What are the compact subsets?

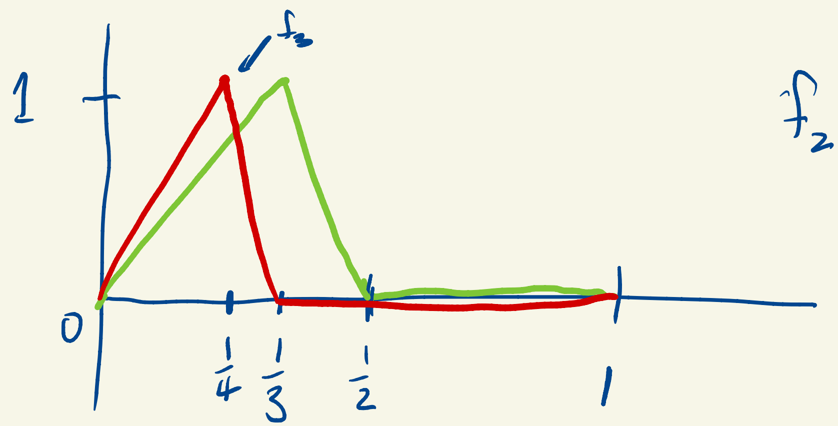
metric space

complete + totally bounded

closed subsets because $C[0,1]$ is complete

totally bounded

f_n



(f_n)

$$\|f_2 - f_3\|_\infty = 1$$

$$\|f_n - f_m\|_\infty = 1$$

$n \neq m$

Recall: a space is totally bounded if every sequence in it has a Cauchy subsequence.

[The sequence just given has no Cauchy subsequence and hence the bounded collection $\{f_n : n \in \mathbb{N}\}$ is not totally bounded. bounded $\not\Rightarrow$ t.b. in $C[0,1]$

Suppose X is a metric space and (x_n) is a sequence in X such that $d(x_n, x_m) \geq m$ for some $m > 0$ and all $n \neq m$.

Then (x_n) has no Cauchy subsequence.

Suppose not and let (x_{n_k}) be Cauchy.

Then there exists K such that if $k, l \geq K$ then $d(x_{n_k}, x_{n_l}) < \frac{m}{2}$. In particular

$$d(x_{n_K}, x_{n_{K+1}}) < \frac{m}{2}$$ which contradicts the fact that $d(x_{n_K}, x_{n_{K+1}}) \geq m$.

Def: Let X be a metric space. A collection $\mathcal{F} \subseteq C(X)$ is <u>equicontinuous</u> if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x,y) < \delta$ and for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$.

If $\mathcal{F}$ has just one function, then this is nothing more than uniform continuity.

If $\mathcal{F}$ is a finite collection of uniformly continuous functions then it is equicontinuous.

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not equicts: $\exists$ bad ε_0 such that $\forall \delta > 0$

there exists $x, z \in X$ and $f \in \mathcal{F}$

with $d(x,z) < \delta$ but $|f(x) - f(z)| \geq \varepsilon_0$.

$\varepsilon_0 = \frac{1}{2}$ $\quad \delta > 0$, n $\quad \left|\frac{1}{n+1} - \frac{1}{n}\right| < \delta$

$$\left|f_n\left(1/_{n+1}\right) - f_n\left(1/_n\right)\right| = 1 > \frac{1}{2}.$$

$\mathrm{Lip}[0,1]$ $\qquad f_m(x) = mx \qquad \varepsilon_0 = 1$

$m \in \mathbb{R}$

$\mathrm{Lip}_K[0,1] \qquad \{ f \in \mathrm{Lip}[0,1] : \text{lip const} \leq K \}$

equicts: Let $\varepsilon > 0$. Let $\delta = \varepsilon / K$

Pick $f \in \mathrm{Lip}_K[0,1]$.

Suppose $x, z \in [0,1]$ $\quad |x - z| < \delta$.

Then $|f(x) - f(z)| \leq K|x - z|$

$< K\delta$

$$= K \frac{\varepsilon}{K} = \varepsilon.$$

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boundedness and equicontinuity are independent $C[0,1]$

$\downarrow$

in norm

$\|\cdot\|_\infty$

not bounded, not e.c.	:	$C[0,1]$
bounded, not e.c	:	$B_1(0)$
not bounded, e.c.	:	$Lip_K[0,1]$
bounded, e.c.	:	$B_1(0) \cap Lip_K[0,1]$

totally bounded $\Rightarrow$ bounded

totally bounded $\Rightarrow$ equicontinuity ($C(X)$ X is cpct.)

Prop: Let X be compact. Every totally bounded subset $\mathcal{F} \subset C(X)$ is equicontinuous.

Pf: Let $\varepsilon > 0$. Let $\{f_1, \dots, f_n\}$ be an ε net for $\mathcal{F}$. The family $\{f_1, \dots, f_n\}$ consists of uniformly continuous functions since X is compact, and since the collection is finite it is equicontinuous. So we can find $\delta > 0$ such that if $x, z \in X$ and $d(x,z) < \delta$ then $d(f_k(x), f_k(z)) < \boxed{\varepsilon}$ for $k = 1, \dots, n$.

Now pick $f \in \mathcal{F}$. There exists f_k such that $\| f - f_k \|_\infty < \boxed{\varepsilon}$. Now suppose $x, z \in X$ and $d(x,z) < \delta$.

Then $$|f(x)-f(z)| \le |f(x)-f_k(x)| + |f_k(x)-f_k(z)| + |f_k(z)-f(z)|$$

$$< \boxed{\varepsilon} + \boxed{\varepsilon} + \boxed{\varepsilon}$$

↳ finitely many k's.

$$= 3\varepsilon. \qquad \square$$

$C(X)$ with X compact

totally bounded $\Rightarrow$ uniformly bounded, equicontinuous

$\Rightarrow$ $\underbrace{\text{pointwise bounded}}$, equicontinuous p.b. $\Rightarrow$ $\forall x \in X$ there exists M (depending on x)

such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$.

$f(x) = x$ on $\mathbb{R}$ $\{f\}$

X space $\quad \mathcal{F} \subseteq C(X)$ is uniformly bound if $\exists M$

such that $|f(x)| \leq M \quad \forall f \in \mathcal{F}$
$\forall x \in X$

$f \in B(X) \quad \| \cdot \|_\infty$