Challenge: pointwise bounded + equicontinuous (X compact)

$\Rightarrow$ uniformly bounded + equicts.

Theorem Arzela-Ascoli Theorem.

Let X be compact. A subset $\mathcal{F} \subseteq C(X)$ is compact if and only if it is

- closed] complete
- pointwise bounded
- equicontinuous] t. b.

Prop: Let X be compact. If $\mathcal{F} \subseteq C(X)$ is pointwise bounded and equicontinuous then it is totally bounded.

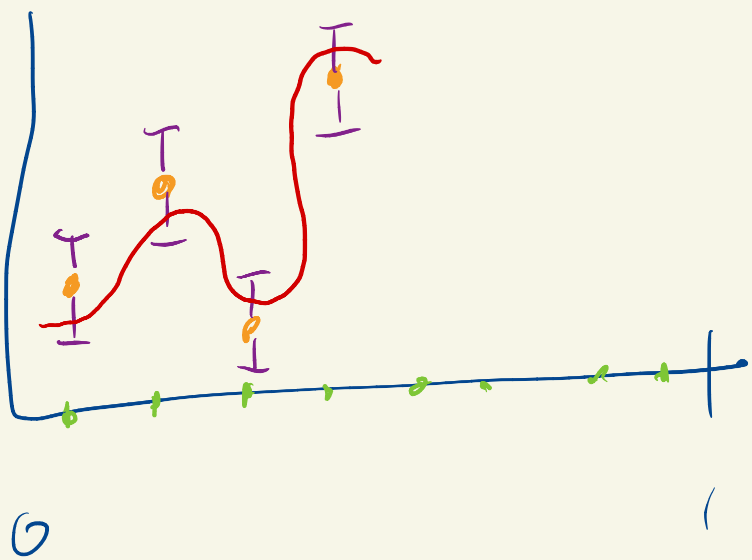
Pf: Suppose $\mathcal{F} \subseteq C(X)$ is pointwise bounded and equicts. Let $\varepsilon > 0$. First, pick $\delta > 0$ such that for all $f \in \mathcal{F}$, if $d(x,z) < \delta$ then $|f(x) - f(z)| < \varepsilon$. Let $x_1, \dots, x_K$ be a δ-net for X. This is possible since X is compact and hence totally bounded.

Pick $M > 0$ such that $|f(x_k)| \leq M$ for all $f \in \mathcal{F}$ and each $1 \leq k \leq K$. This is possible since $\mathcal{F}$ is pointwise bounded.

Now let $y_1, \dots, y_J$ be an ε net for $[-M, M]$.

Let P be the set of functions from $\{x_1, \dots, x_K\}$ to $\{y_1, \dots, y_J\}$. There are J^K such functions.

Given $p \in P$ let $\mathcal{F}_p = \{ f \in \mathcal{F} : |f(x_k) - p(x_k)| < \varepsilon$
$1 \le k \le K \}$.



I claim that

$$\bigcup_{p \in P} \mathcal{F}_p = \mathcal{F}.$$

Indeed, if $f \in \mathcal{F}$ then each $f(x_k) \in [-M, M]$

and we can pick y_k's with $|f(x_k) - y_k| < \varepsilon$.

Setting $p(x_k) = y_k$ for each k, $f \in \mathcal{F}_p$.

We claim each diam $\mathcal{F}_p \leq 4\varepsilon$, in which case we have shown

$\mathcal{F}$ is t. b. Suppose $f, g \in \mathcal{F}_p$. Consider $x \in X$.

Then there exists x_k with $d(x, x_k) < \delta$.

Then
$$|f(x) - g(x)| \leq |f(x) - f(x_k)| + |f(x_k) - p(x_k)| + |p(x_k) - g(x_k)| + |g(x_k) - g(x)|.$$

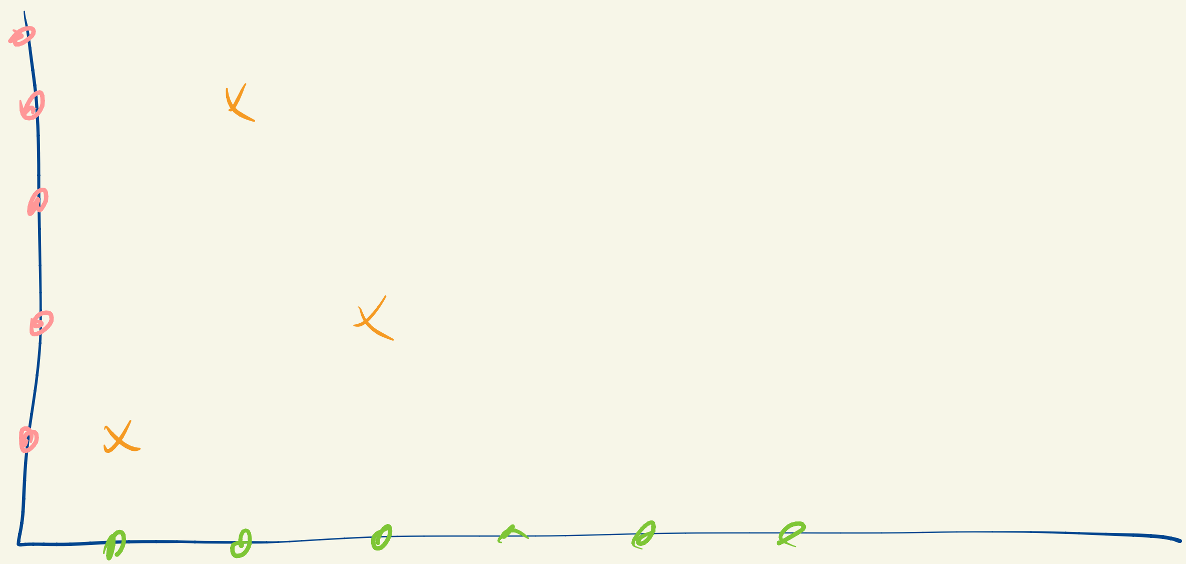
$$< \underset{\text{e.c.}}{\varepsilon} + \underset{f \in \mathcal{F}_p}{\varepsilon} + \underset{g \in \mathcal{F}_p}{\varepsilon} + \underset{g \in \mathcal{F}\ \text{e.c.}}{\varepsilon}$$

$$= 4\varepsilon.$$

Hence $\|f-g\| \le 4\varepsilon$ and therefore $\operatorname{diam} \mathcal{F}_p \le 4\varepsilon$.

$\square$

These $\mathcal{F}_p$'s will be empty!



Recall: $A \subseteq X$ is t.b. if for every $\varepsilon > 0$

there exist finitely many $A_k \subseteq X$ such that

1) $\operatorname{diam} A_k \leq \varepsilon$

2) $\bigcup A_k \supseteq A$.

x_k

$B_{2\varepsilon}(x_k) \supseteq A_k$

$\bigcup B_{2\varepsilon}(x_k) \supseteq \bigcup A_k \supseteq A$.

Integration

Riemann Integral

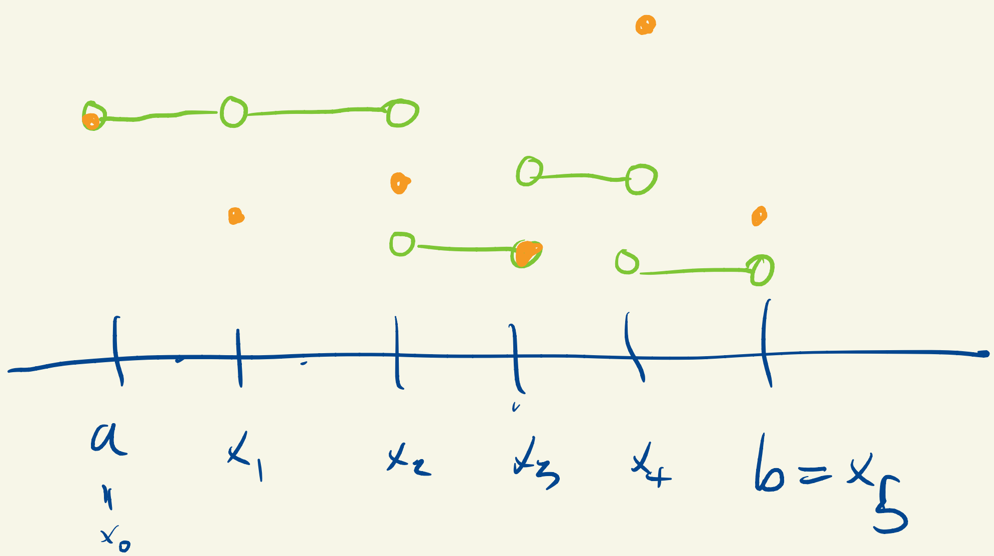
$[a,b] \subseteq \mathbb{R} \quad a < b$

Partitions: $a = x_0 < x_1 < \cdots < x_n = b$ $\quad \{x_k\}_{k=0}^{n} \leftarrow$ partition

(its a finite subset of $[a,b]$ that includes a, b).

Step functions: $\text{Step}[a,b]$

$g \in \text{Step}[a,b]$ if there exists a partition P of $[a,b]$ such that g is constant on each (x_{k-1}, x_k) $k = 1, \ldots, n$.



We call such a partition a <u>step partition</u> for g.

We'd like to define $\int_a^b g$ if $g \in$ Step.

1) pick a step partition P for g.

$$a = x_0 < x_1 < \cdots < x_n = b$$

2) Let $dx_k = x_k - x_{k-1} \quad k = 1, \ldots, n$

3) $\displaystyle {}^{\mathcal{P}}\!\!\int_a^b g = \sum_{k=1}^{n} g_k \, dx_k$ where g_k is the constant value of g on (x_{k-1}, x_k).

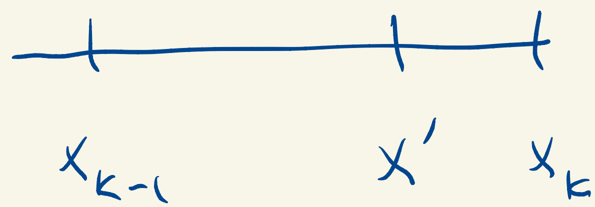
We need to show that the value ${}^{\mathcal{P}}\!\!\int_a^b g$ is independent of the step partition $\mathcal{P}$ and hence we can define $\int_a^b g = {}^{\mathcal{P}}\!\!\int_a^b g$

We say $\mathcal{P}'$ is a <u>refinement</u> of $\mathcal{P}$ if $\mathcal{P}' \supseteq \mathcal{P}$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions we call $\mathcal{P}_1 \cup \mathcal{P}_2$ the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$.

Exercise: Suppose $g \in \text{Step}[a,b]$ and $\mathcal{P}$ is a step partition for g. If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ with exactly one additional point then

$$\overset{\mathcal{P}'}{\int_a^b} g = \overset{\mathcal{P}}{\int_a^b} g$$



Cor: If $\mathcal{P}'$ is a refinement of a step partition for $g \in \text{Step}[a,b]$ then

$$\overset{\mathcal{P}'}{\int_a^b} g = \overset{\mathcal{P}}{\int_a^b} g.$$

Cor: If $\mathcal{P}_1$ and $\mathcal{P}_2$ are step partitions for $g \in \text{Step}[a,b]$ then

$$\overset{\mathcal{P}_1}{\int_a^b} g = \overset{\mathcal{P}_2}{\int_a^b} g.$$

$$\int_a^{b} {}^{P_1} g = \int_a^{b} {}^{P_2} g$$

$$\int_a^{b} {}^{P_1 \cup P_2} g$$

Properties: 1) Linearity $\int_a^b g_1 + g_2 = \int_a^b g_1 + \int_a^b g_2 \quad g_1, g_2 \in \text{Step}[a,b]$

$$\int_a^b cg = c \int_a^b g$$

2) Monotonicity: $g_1 \leq g_2 \Rightarrow \int_a^b g_1 \leq \int_a^b g_2$

3) $\left| \int_a^b g \right| \leq \int_a^b |g|$

4) If $a < c < b$

$$\int_a^b g = \int_a^c g + \int_c^b g$$

(If $g \in Step[a,b]$ then $g|_{[a,c]} \in Step[a,c]$

$g|_{[c,b]} \in Step[c,b]$)