Monotonicity

$g_1, g_2 \in \mathrm{Step}[a,b]$ $\qquad g_1 \leq g_2$

Use a step partition for both.

$$\int_a^b g_1 = \sum_{k=1}^{n} g_{1,k} \Delta x_k \qquad g_{1,k} \leq g_{2,k}$$

$$\int_a^b g_2 = \sum_{k=1}^{n} g_{2,k} \Delta x_k \qquad g_1 \leq g_2$$

$$\Rightarrow \quad \sum_{k=1}^{n} g_{1,k} \Delta x_k \leq \sum_{k=1}^{n} g_{2,k} \Delta x_k$$

$$g \in \mathrm{Step}[a,b] \Rightarrow |g| \in \mathrm{Step}[a,b]$$

$$-|g| \leq g \leq |g|$$

$$\int_a^b -|g| \leq \int_a^b g \leq \int_a^b |g|$$

$$-\int_a^b |g| \leq \int_a^b g \leq \int_a^b |g|$$

$$\left| \int_a^b g \right| \leq \int_a^b |g|$$

$B[a,b]$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$\notin B[0,1]$

$f \in B[a,b]$ $\quad g \leq f \leq G$

$g, G \in \mathrm{Step}[a,b]$ we'd like $\int_a^b g \leq \int_a^b f \leq \int_a^b G$

$f \in B[a,b]$

$$\overline{\int_a^b} f = \inf_{\substack{G \in \mathrm{Step}[a,b] \\ G \geq f}} \int_a^b G \qquad \underline{\int_a^b} f = \sup_{\substack{g \in \mathrm{Step}[a,b] \\ f \geq g}} \int_a^b g$$

If $\overline{\int_a^b} f = \underline{\int_a^b} f$ we say that f is $\underbrace{\text{Riemann integrable}}_{\Downarrow}$

and we define $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$.

$R[a,b] \subseteq B[0,1]$.

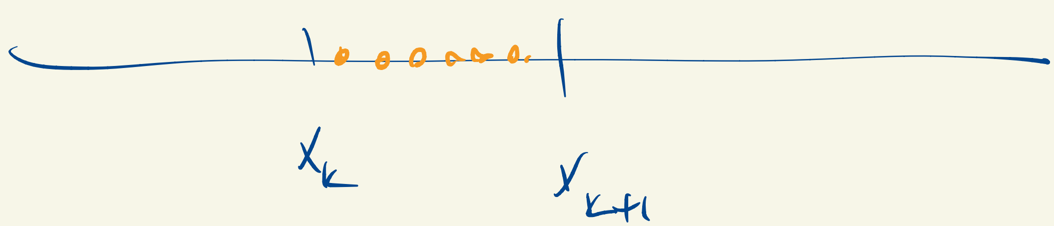
Exercise: $\chi_{\mathbb{Q}} = \begin{cases} 1 & \text{on } \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$

$\in B[0,1]$

$\notin R[0,1]$

$\overline{\int_0^1} \chi_{\mathbb{Q}} = 1$

$\underline{\int_0^1} \chi_{\mathbb{Q}} = 0$



Prop: Suppose $f \in B[a,b]$.

Then $f \in R[a,b]$ iff for every $\varepsilon > 0$

there exist $g, G \in Step[a,b]$ with

$$g \le f \le G$$

and $$\int_a^b G - g < \varepsilon.$$

Pf: Suppose $f \in R[a,b]$. Let $\varepsilon > 0$. Find $G \in Step[a,b]$, $G \ge f$,

with $$\int_a^b G < \overline{\int_a^b} f + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}.$$

Similarly, find $g \in Step[a,b]$, $g \le f$, and

$$\int_a^b g > \underline{\int_a^b} f - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2}.$$

Subtracting we find

$$\int_a^b G - \int_a^b g < \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

That is $\displaystyle\int_a^b G-g < \varepsilon.$

Conversely, suppose $f \in B[a,b]$ and $f \notin \mathcal{R}[a,b]$.

Let $\varepsilon = \overline{\int_a^b} f - \underline{\int_a^b} f > 0$ since $f \notin \mathcal{R}[a,b]$.

If $G \in Step[a,b]$ with $G \geq f$ then

$$\int_a^b G \geq \overline{\int_a^b} f.$$

if $g \in Step[a,b]$ with $g \leq f$ then

$$\int_a^b g \leq \underline{\int_a^b} f$$

So $\int_a^b G - g = \int_a^b G - \int_a^b g \geq \overline{\int_a^b} f - \underline{\int_a^b} f = \varepsilon.$

Is $Step[a,b] \subseteq R[a,b]$?

$f \in Step[a,b]$ If $G \in Step[a,b]$

$\underline{\underline{G \geq f}}$

$$\hat{\int_a^b} f \leq \underline{\underline{\hat{\int_a^b} G}}$$

$$\hat{\int_a^b} f \leq \overline{\int_a^b} f$$

$\hat{\int_a^b} f \geq \overline{\int_a^b} f$ since $f \in Step[a,b]$ $f \geq f$.

$$\boxed{\hat{\int}_a^b = \overline{\int}_a^b f}$$ and now similarly for $\underline{\int}_a^b f$.

$$\overline{\int}_a^b f = \underline{\int}_a^b f = \hat{\int}_a^b f \Rightarrow f \in R[a,b]$$

$$\int_a^b f = \hat{\int}_a^b f$$

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cpct!

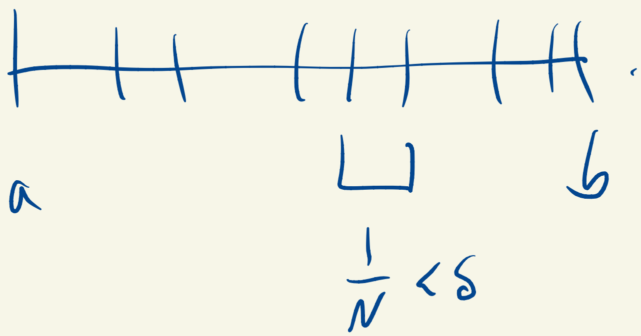
Prop: $C[a,b] \subseteq R[a,b]$.

Pf: Let $f \in C[a,b]$. Let $\varepsilon > 0$.

Pick $\delta > 0$ such that if $x, z \in [a,b]$ with $|x - z| < \delta$ then $|f(x) - f(z)| < \frac{\varepsilon}{b-a}$. This is possible since f is uniformly continuous.

Pick $N \in \mathbb{N}$ with $\frac{1}{N} < \delta$.

Let $x_k = a + k\Delta x$ with $\Delta x = \frac{b-a}{N}$, $0 \le k \le N$.



On interval $I_k = [x_{k-1}, x_k]$

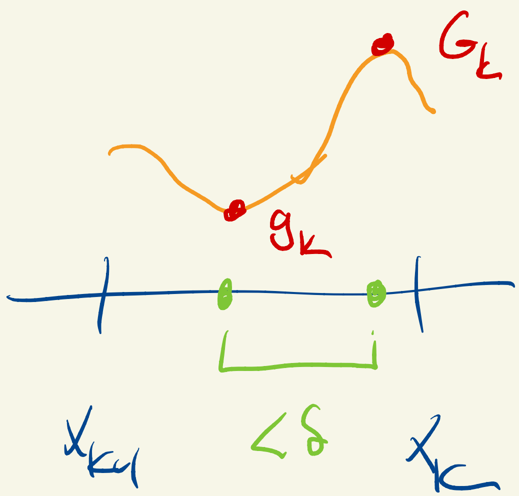
let $G_k = \max\limits_{x \in I_k} f(x)$

$g_k = \min\limits_{x \in I_k} f(x)$.

Let G be the step function that equals G_k on each (x_{k-1}, x_k) and equals f on each x_k.

So $G \geq f$. Define $g \leq f$ similarly.

Observe that for each k $\quad G_k - g_k < \frac{\varepsilon}{b-a}$.



Then

$$\int_a^b G - g = \sum_{k=1}^{n} (G_k - g_k)\,\Delta x$$

$$< \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta x$$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

$\square$

Properties:

1) Linearity

If $f, g \in \mathcal{R}[a,b]$ then $\underline{f+g \in \mathcal{R}[a,b]}$

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

Similarly for scaler mult.

2) Monotonicity: if $f, g \in \mathcal{R}[a,b]$ and $f \leq g$

then $\int_a^b f \leq \int_a^b g$

3) If $f \in \mathcal{R}[a,b]$ then $\underline{|f| \in \mathcal{R}[a,b]}$ and

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

4) If $f \in \mathcal{R}[a,b]$ and $a < c < b$ then

$f|_{[a,c]} \in \mathcal{R}[a,c]$ and $f|_{[c,b]} \in \mathcal{R}[c,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

1) Linearity

If $f, g \in \mathcal{R}[a,b]$ then $f+g \in \mathcal{R}[a,b]$

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

We can find step functions $H_{f,n}$ with $H_{f,n} \geq f$

and $\int_a^b H_{f,n} \leq \int_a^b f + \frac{1}{n}$. Note: $\int_a^b H_{f,n} \to \int_a^b f$

Similarly we can find step functions $h_{f,n}$ with $h_{f,n} \leq f$

and $\int_a^b h_{f,n} \geq \int_a^b f - \frac{1}{n}$ so $\int_a^b h_{f,n} \to \int_a^b f$.

Find similar step functions $H_{g,n}$ and $h_{g,n}$ for g.

Now

$$\int_a^b h_{f,n} + h_{g,n} \le \underline{\int_a^b} f+g \le \overline{\int_a^b} f+g \le \int_a^b H_{f,n} + H_{g,n}$$

$$\int_a^b h_{f,n} + h_{g,n} = \int_a^b h_{f,n} + \int_a^b h_{g,n}, \qquad \int_a^b H_{f,n} + H_{g,n} = \int_a^b H_{f,n} + \int_a^b H_{g,n}$$

for all n.

Taking a limit in n we find

$$\int_a^b f + \int_a^b g \le \underline{\int_a^b} f+g \le \overline{\int_a^b} f+g \le \int_a^b f + \int_a^b g.$$

Hence all inequalities above are equalities and

$$\overline{\int_a^b} f+g = \underline{\int_a^b} f+g = \int_a^b f + \int_a^b g.$$

So $f+g \in \mathcal{R}[a,b]$ and $\int_a^b f+g = \int_a^b f + \int_a^b g$. $\square$

$$\int_{-\pi}^{\pi} \cos(3x)\cos(19x)\,dx = 0$$

3.1

$$\frac{1}{42}\int_{-\pi}^{\pi} h \cdot \sin(kx)\,dx = 0$$

$k = 0, \dots, \infty$

$$\frac{1}{42}\int_{-\pi}^{\pi} h \cos(kx)\,dx = 0$$

j l

$\Rightarrow h = 0$ (!)

$f \geq 0$

$$\boxed{\int_a^b f = 0 \ + \ f \geq 0 \ + \ f \text{ is cts} \ \Rightarrow f = 0}$$