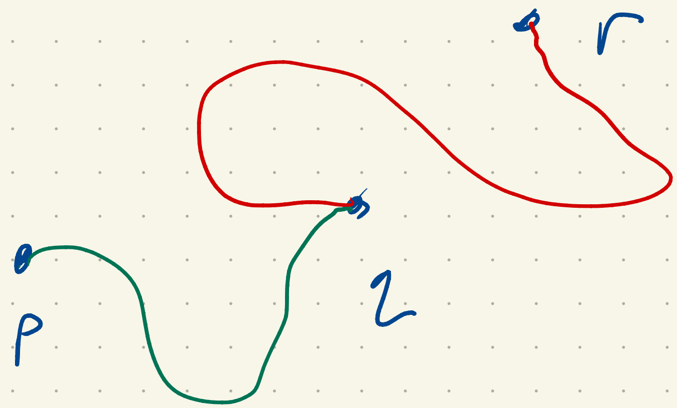
open in $[0,1]$ and closed in $[0,1]$.

$\Rightarrow \gamma$ is const.

Exercise: The continuous image of a path connected space is path connected.

Exercise: If $p, q, r \in X$ and there is a path from p to q and a path from q to r then there is also a path from p to r.



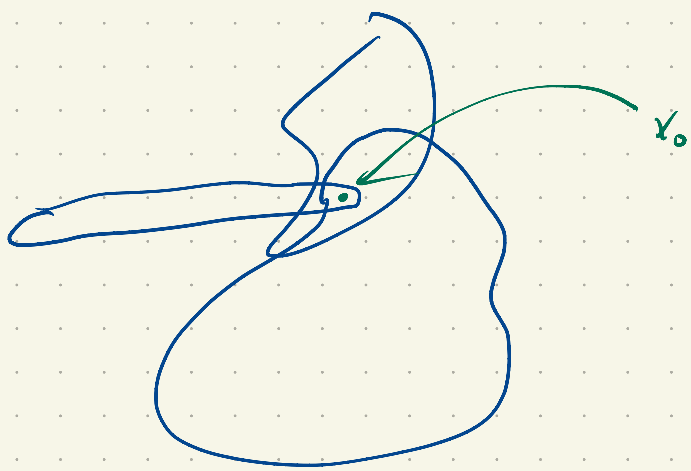
$\gamma_1 : [0,1] \to X$

$\gamma_2 : [0,1] \to X$

$\hat{\gamma}_1 : [0,1/2] \to X$

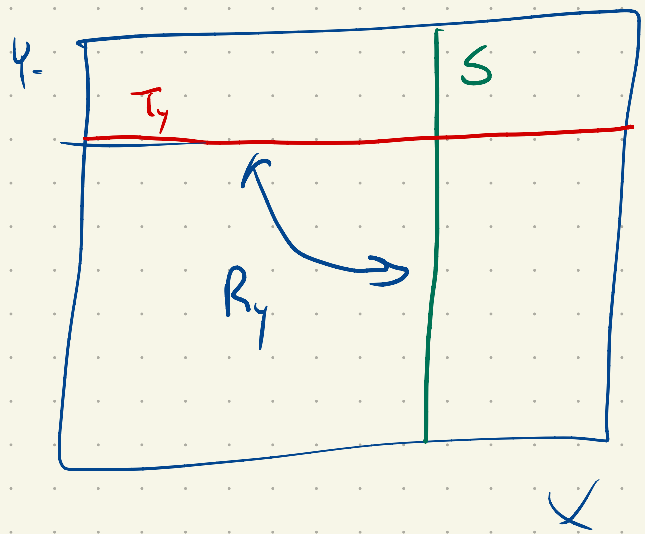
$\hat{\gamma}_2 : [1/2,1] \to X$

Exercise: If $\{X_\alpha\}_{\alpha \in I}$ is a collection of path connected sets and $\bigcap X_\alpha \neq \emptyset$ then $\bigcup X_\alpha$ is path connected.

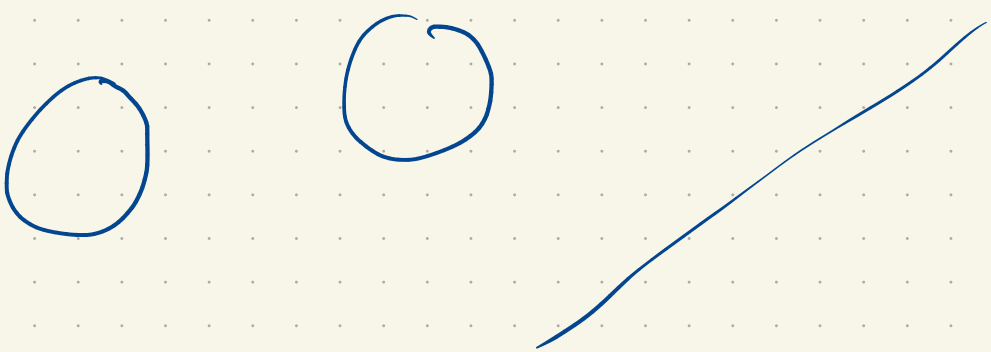


Prop: A quotient of a path connected space is path connected.

Prop: A finite product of path connected spaces is path connected.



$$X \times Y = \bigcup_y R_y \qquad \bigcap R_y \supseteq S \neq \emptyset$$



Components

Let $x \in X$. Let $\mathcal{A}_x = \{A \subseteq X : x \in A, A \text{ is connected}\}$

Let $\text{Comp}(x) = \bigcup_{A \in \mathcal{A}_x} A$. $\bigcap_{A \in \mathcal{A}_x} A \supseteq \{x\} \neq \emptyset$

Observe $\text{Comp}(x)$, the (connected) component of x is connected.

$\text{Comp}(x) \in \mathcal{A}_x$. It's distinguished because it contains every element of $\mathcal{A}_x$. It is the largest element.

We could have defined $\text{Comp}(x)$ to be the largest connected set that contains x.

Exercise: If $A \subseteq \mathbb{Q}$ has more than one point then A is disconnected.

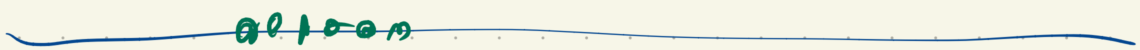
$[0,1] \cap \mathbb{Q}$

$(0,1) \cap \mathbb{Q}$

Hence if $q \in \mathbb{Q}$ $\quad \mathrm{Comp}(q) = \{q\}$

Such a space is called totally disconnected.

Note: this is not the same as the discrete top.



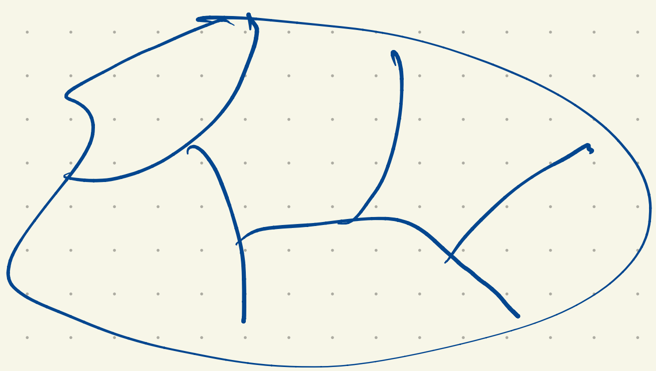
Suppose $x, y \in X$. What can we say about $\mathrm{Comp}(x)$ vs $\mathrm{Comp}(y)$.

Then either $\mathrm{Comp}(x) \cap \mathrm{Comp}(y) = \emptyset$ or

$$\mathrm{Comp}(x) = \mathrm{Comp}(y)$$

If $\mathrm{Comp}(x) \cap \mathrm{Comp}(y) \neq \emptyset$ then $\mathrm{Comp}(x) \cup \mathrm{Comp}(y)$ is connected and contains x and y. So $\mathrm{Comp}(x) \supseteq \mathrm{Comp}(x) \cup \mathrm{Comp}(y)$ and $\mathrm{Comp}(x) \supseteq \mathrm{Comp}(y)$ and vice-versa.

The components of a space partition it



$x \sim y$ if there is a connected set containing x, y.

A connected space has one component. $\mathrm{Comp}(x) = X$.

$A \subseteq B \subseteq \overline{A}$

connected

connected!

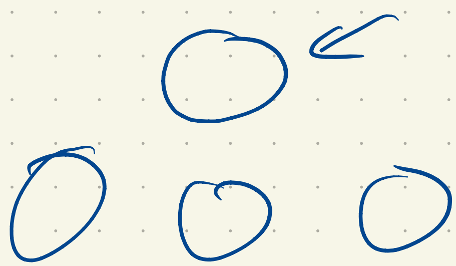
Consider a component $\mathrm{Comp}(x)$.

Observe that $\overline{\mathrm{Comp}(x)}$ is connected and contains x.

So $\mathrm{Comp}(x) \supseteq \overline{\mathrm{Comp}(x)}$ and have $\mathrm{Comp}(x)$ is closed.

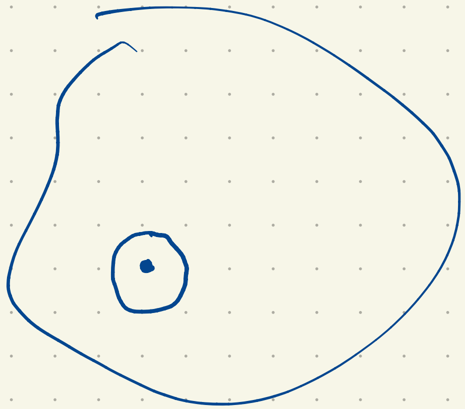
Are components necessarily open? No! In $\mathbb{Q}$ the components are singletons that are not open.

If a space has finitely many components then the components are open.



(components are a finite intersection of open sets)

Def: A space is locally connected if it admits a basis of connected open sets.



Exercise: If C is a component of X and $x \in C$ then

$\mathrm{Comp}(x) = C$.

Prop: In a locally connected space components are open

Pf: Consider a component C. Let $x \in C$ and find a connected basic open set B containing x. Since C is the largest connected set containing x, $x \in B \subseteq C$. Hence C is open.

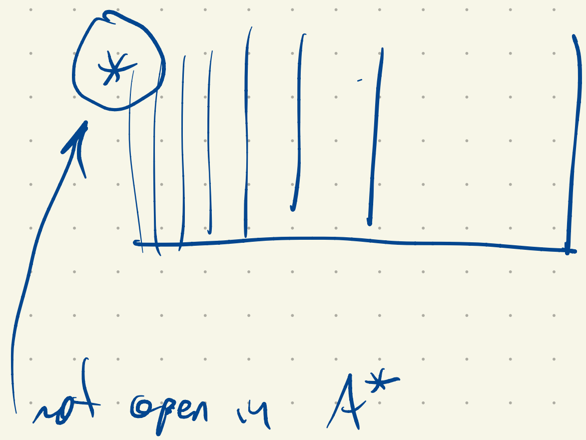
Note: manifolds are locally connected.

There are path connected variants of the above.

$PComp(x)$ is the largest path connected set containing x.

It is the union of all path connected sets containing x.

It need not be open or closed.



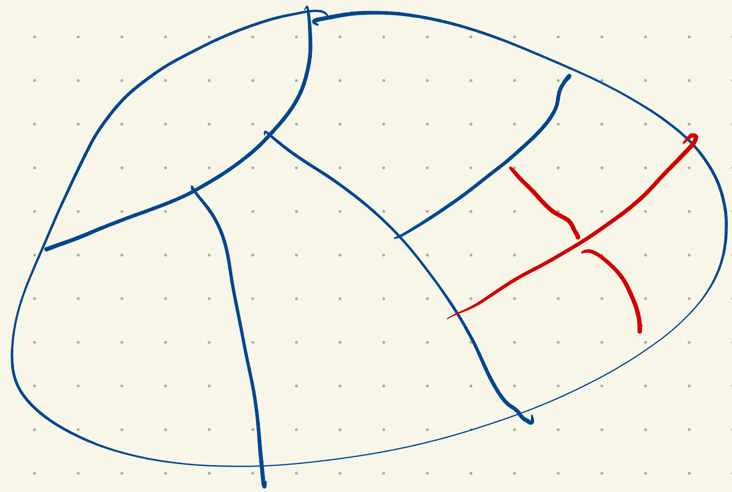
Just one component.

$\overline{A}$ in A^* is A^*

The path components of a space partition the space.

Each path component is contained in a single component.

$x \in PComp(x) \subseteq Comp(x)$



Locally path connected: Has a basis of path connected sets.

Exercise: Path components in a locally path connected space are open and closed.

Prop: In a locally path connected space the components and path components coincide.

Pf: Let $x \in X$ and consider $\mathrm{Comp}(x)$ and $P\mathrm{Comp}(x)$.

Since $PComp(x)$ is both open and closed in X and is contained in $Comp(x)$ it is both open and closed in $Comp(x)$. Since $Comp(x)$ is connected and since $PComp(x) \neq \emptyset$, $PComp(x) = Comp(x)$.

Cor: A locally p.c. space is connected $\Leftrightarrow$ it is path connected,

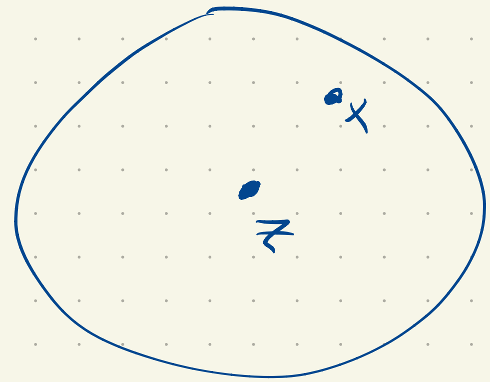
Cor: Connected manifolds are exactly the path connected manifolds

(Manifolds are locally path connected)

Why are balls in $\mathbb{R}^n$ path connected?

$B_r(z)$ is is path connected.

Consider $x \in B_r(z)$.

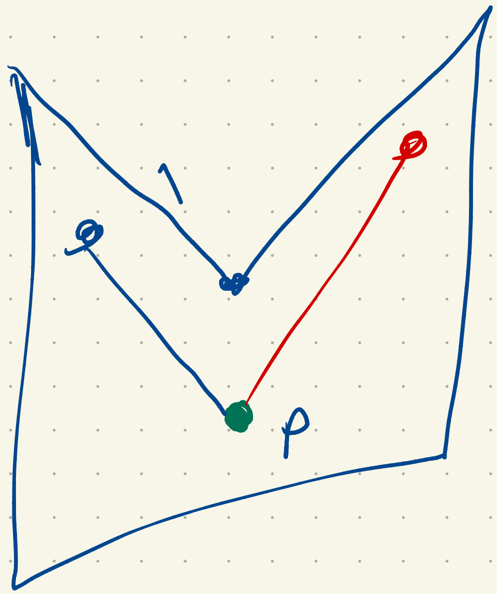


$\gamma(s) = (1-s)z + sx$

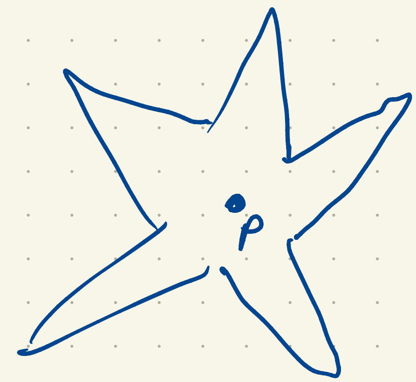
$0 \leq s \leq 1$

$$d(z, (1-s)z + sx) = \|(1-s)z + sx - z\|$$
$$= \|s(x-z)\|$$
$$= |s|\,\|x-z\|$$
$$= |s|\, d(z,x)$$
$$< |s|\, r$$
$$\leq r.$$

$(1-s)z_k + s x_k$



Star shaped w.r.t. p.



These are path connected.