Compactness

Def: A top space X is <u>compact</u> if whenever $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets with $X = \bigcup_{\alpha \in I} U_\alpha$, there exist finitely many $U_{\alpha_1}, \dots, U_{\alpha_k}$ such that

$$X = \bigcup_{j=1}^{k} U_{\alpha_j}.$$

open cover

admits a finite subcover

Compactness is topological.

Better than that:

Prop: Suppose $f: X \to Y$ is continuous and surjective. If X is compact then so is Y.

Pf: Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of Y. The sets $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ are an open cover of X.

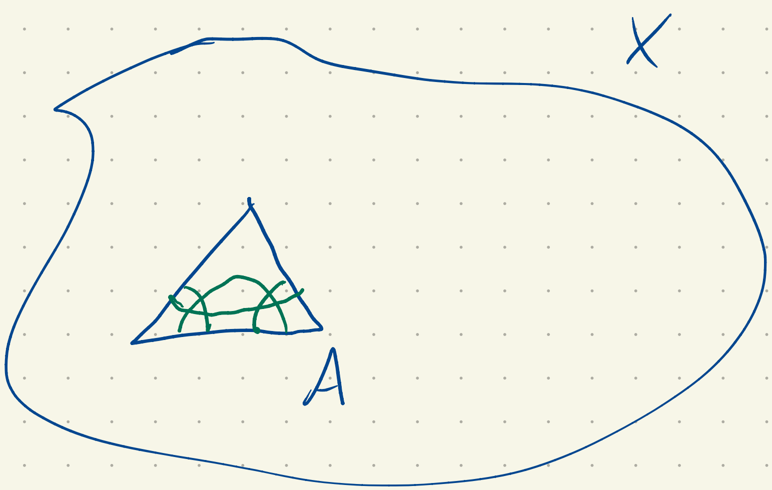
Now reduce to a finite subcover $\{f^{-1}(U_{\alpha_j})\}_{j=1}^{n}$.

Observe $\displaystyle Y = f(X) = f\left(\bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j})\right)$

$$= \bigcup_{j=1}^{n} f\left(f^{-1}(U_{\alpha_j})\right)$$

$$= \bigcup_{j=1}^{n} U_{\alpha_j} \quad \text{(by surjectivity).}$$

Cor: If $f: X \to Y$ is a homeomorphism, X is compact iff Y is compact.



A is compact means with respect to the subspace topology.

We can detect compactness of A using open sets in X only.

Def: A collection of open sets $\{U_\alpha\}_{\alpha \in I}$ in X is an <u>open cover</u> of $A \subseteq X$ if $A \subseteq \bigcup_{\alpha \in I} U_\alpha$.

It admits a <u>finite subcover</u> if there are finitely many U_{α_j} $j=1,\dots,n$ with $A \subseteq \bigcup_{j=1}^{n} U_{\alpha_j}$.

Prop: A set $A \subseteq X$ is compact iff every open cover of A by sets in X admits a finite subcover.

Pf. Suppose A is compact and $\{U_\alpha\}_{\alpha \in I}$ is an open cover of A by sets in X. Then $\{A \cap U_\alpha\}_{\alpha \in I}$ is an intrinsic open cover of A that admits a finite subcover $A \cap U_{\alpha_j}$ $j=1,\dots,n$.

But then $\bigcup_{j=1}^{n} U_{\alpha_j} \supseteq \bigcup_{j=1}^{n} (U_{\alpha_j} \cap A) = A$.

Conversely suppose every open cover of A by sets in X admits a finite subcover. Consider an intrinsic open cover $\{V_\alpha\}_{\alpha\in I}$ of A by open sets in A. Each $V_\alpha = A \cap U_\alpha$ for some U_α open in X. The sets $\{U_\alpha\}_{\alpha\in I}$ are an open cover of A by sets in X. It admits a finite subcover U_{α_j} $j=1,\dots,n$. We claim $\{V_{\alpha_j}\}_{j=1}^n$ covers A.

Indeed,

$$A \subseteq \bigcup_{j=1}^n U_{\alpha_j} \Rightarrow A = A\cap A \subseteq \bigcup_{j=1}^n (U_{\alpha_j}\cap A) = \bigcup_{j=1}^n V_{\alpha_j}.$$

E.g. $(0,1] \subseteq \mathbb{R}$ is not compact.

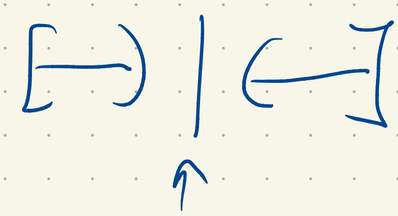
$\{(\frac{1}{n},1]\}_{n\in\mathbb{N}}$ is an open cover by sets in $(0,1]$ with no finite subcover.

$\{(\frac{1}{n},\infty)\}_{n\in\mathbb{N}}$ is an open cover by sets in $\mathbb{R}$ with no finite subcover.

Singletons are compact.

Finite sets are compact.

$[0,1]\cap\mathbb{Q}$ is not compact.



Prop: $[0,1]$ is compact.

Pf: Let $\{U_\alpha\}$ be an open cover of $[0,1]$ by sets in $\mathbb{R}$.

Let $C = \{c\in[0,1] : [0,c] \text{ admits a finite subcover}\}$.

We'll show that C is open, closed and nonempty and hence, since $[0,1]$ is connected, is all of $[0,1]$. In particular, $1\in C$ and therefore $[0,1]$ admits a finite subcover.

Observe C is nonempty as $0\in C$. Suppose d is a contact point of C. Pick U_α from the collection with $d\in U_\alpha$. (Such a choice is possible since $\{U_\alpha\}$ covers $[0,1]$). Since U_α is open, and since d is a contact point, it contains some $c\in C$. If $c>d$ then certainly $[0,d]\subseteq[0,c]$

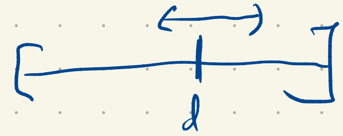
$d-\varepsilon<c<d$

$(d-\varepsilon,d+\varepsilon)\subseteq U_\alpha$

U_α

WLOG we can assume $[c,d]\subseteq U_\alpha$ if $c<d$.

is covered by finitely many U_α's. Otherwise,



$[0,c]$ has a finite subcover and $[c,d]$ is covered by one additional U_α. So $[0,d]$ is covered by finitely many U_α's. Hence $d \in C$ and C is closed.

Now, to show C is open, pick $c \in C$. Let U_α be a set from the cover containing c. Find an interval $(c-\varepsilon, c+\varepsilon) \subseteq U_\alpha$. Each $c' \in (c-\varepsilon, c+\varepsilon) \cap [0,1]$ belongs to C. Indeed, if $c' \leq c$, $[0,c']$ obviously admits a finite subcover (the one for $[0,c]$ works) and if $c' > c$ then we can cover $[0,c']$ with the finitely many sets U_α that cover $[0,c]$ plus one additional set.

$\square$

Prop: If K is compact and $V \subseteq K$ is closed then V is compact.

Pf: Let $\{U_\alpha\}$ be an open cover of V by sets in K.

Observe that V^c is open and $\{U_\alpha\} \cup \{V^c\}$ is an open cover of K. It admits a finite subcover $U_{\alpha_1}, \dots, U_{\alpha_n}, V^c$. But then $\{U_{\alpha_j}\}_{j=1}^{n}$ covers V.

Closed subsets of $[0,1]$ are compact.

$[a,b]$

Prop: In Hausdorff spaces, compact sets are closed.

Pf: Let X be Hausdorff and suppose $V \subseteq X$ is compact.

Let $x \in V^c$. For each $y \in V$ find ~~dis~~ disjoint open sets U_y, W_y with $y \in U_y$ $x \in W_y$.

The sets $\{U_y\}_{y \in V}$ cover V and we extract a finite subcover $\{U_{y_j}\}_{j=1}^n$. Let $W = \bigcap_{j=1}^n W_{y_j}$.

Then I claim $W \cap V = \phi$. Since $x \in W$ we can conclude V^c is open. Note

$$W \cap V \subseteq W \cap \bigcup U_{y_j} = \bigcup_{j=1}^n U_{y_j} \cap W$$

$$\subseteq \bigcup_{j=1}^n U_{y_j} \cap W_{y_j}$$

$$= \phi.$$