Last class: compactness.

**Def:** Let $X$ be a metric space. A set $A \subseteq X$ is <u>bounded</u> if there exists $x \in X$ and $r > 0$ such that $A \subseteq B_r(x)$.

Note: there is complete flexibility in $x$. ($\Delta$ inequality)

**Lemma:** Compact subsets of a metric space are bounded.

**Pf:** Suppose $A \subseteq X$ is not bounded.

Pick $x \in X$ and form the cover $\{B_r(x)\}_{r>0}$

These sets cover $A$ but because $A$ is not bounded there is no finite subcover. So $A$ is not compact.

Cor: In metric spaces compact sets are closed and bounded.

The converse (even for metric spaces) is false.

e.g. $\mathbb{Z}$ $\quad d(x,y) = 1 \quad x \neq y$.

Is $\mathbb{Z}$ bounded? Yep!

Is $\mathbb{Z}$ closed in $\mathbb{Z}$? Yep!

Is $\mathbb{Z}$ compact? No: the open cover by singletons has no finite subcover.

Next semester: compact $\Leftrightarrow$ complete + totally bounded in metric spaces

But! In $\mathbb{R}^n$, compact $\Leftrightarrow$ closed + bounded holds.

(Extreme Value Theorem) Using: subsets of $\mathbb{R}$ are compact $\iff$ closed + bounded.

Prop: Suppose $f : K \to \mathbb{R}$ where $K$ is compact and nonempty, and $f$ is cts.

Then there exist $k_-, k_+ \in K$ such that

$$f(k_-) \le f(k) \le f(k_+) \quad \text{for all } k \in K.$$

Pf: Observe $f(K) \subseteq \mathbb{R}$ is compact. Then $f(K)$ is closed and bounded. Since it is nonempty (as $K$ is nonempty) and bounded above, $A$ admits a supremum $s$. There is a sequence in $f(K)$ converging to $s$ and hence $s \in \overline{f(K)} = f(K)$. Hence there is $k_+ \in K$ with $f(k_+) = s$. So $f(k_+) \geq f(k)$ for all $k \in K$ as $s$ is an upper bound for $f(K)$. The construction of $k_-$ is similar.

$A \subseteq \mathbb{R}$, closed, bounded

bounded $\Rightarrow$ $A \subseteq [-R, R]$ for some $R$.

$[-R, R] \sim [0,1]$ which is compact.

$A$ is closed in $\mathbb{R}$ and contained in $[-R, R]$ and is hence closed in $[-R, R]$.

Hence $A$ is compact as a subspace of $[-R, R]$ but then also as a subspace of $\mathbb{R}$.

How to generalize this reasoning to $\mathbb{R}^n$?

Steps:

1) $[-R, R]$ is compact for all $R > 0$.

2) If $X_1, \dots, X_n$ are compact then $X_1 \times \dots \times X_n$ is compact.

3) $[-R, R]^n$ is compact w/ product top

but then also as a subspace of $\mathbb{R}^n$.

4) If $A \subseteq \mathbb{R}^n$ is bounded then

$A \subseteq [-R, R]^n$ for some large $R$.

$B_r(0)$ $\qquad x = (x_1, \dots, x_n)$

↳ $\subseteq [-r, r]^n$.

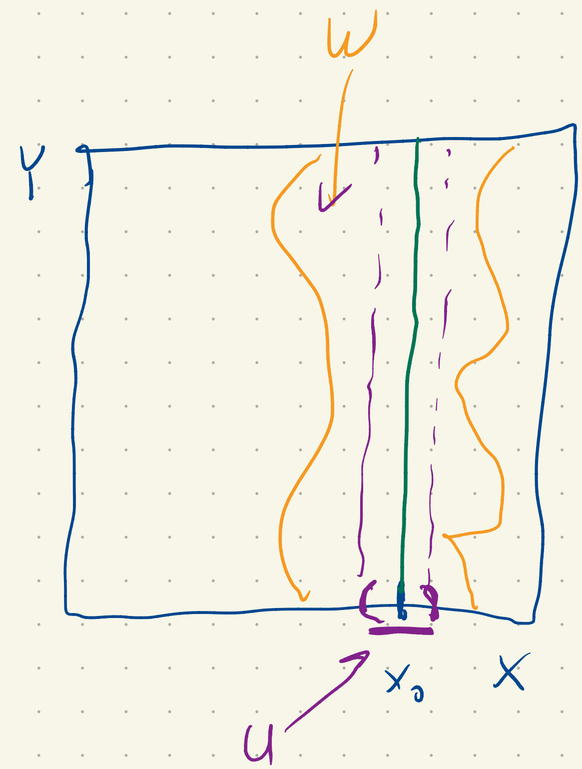
5) If in addition $A$ is closed then

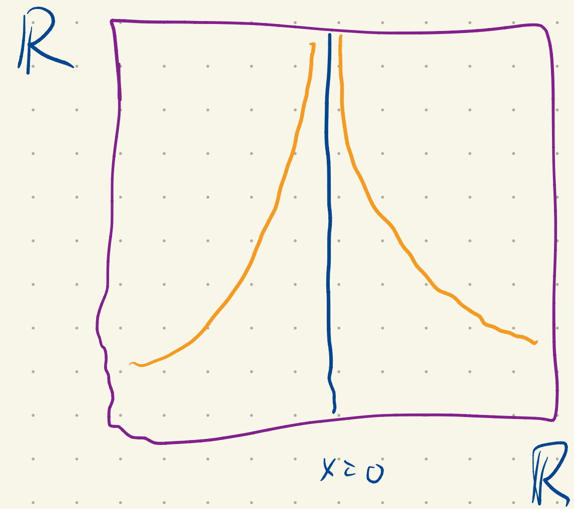
the above argument shows $A$ is compact.

**Lemma:** (Tube Lemma)

Suppose $X, Y$ are spaces and $Y$ is compact. Let $x_0 \in X$ and let $W$ be an open set containing $\{x_0\} \times Y$. Then there is an open set $U \subseteq X$ with $x_0 \in U$ and

$$\underbrace{U \times Y}_{\pi_X^{-1}(U)} \subseteq W.$$




**Pf:** For each $(x_0, y)$ we can find a basic open set $U_y \times V_y \subseteq W$ with $x_0 \in U_y$ and $y \in V_y$. The fiber $\{x_0\} \times Y$ is homeomorphic to $Y$ and hence compact.

Then sets $U_y \times V_y$ cover the fiber,

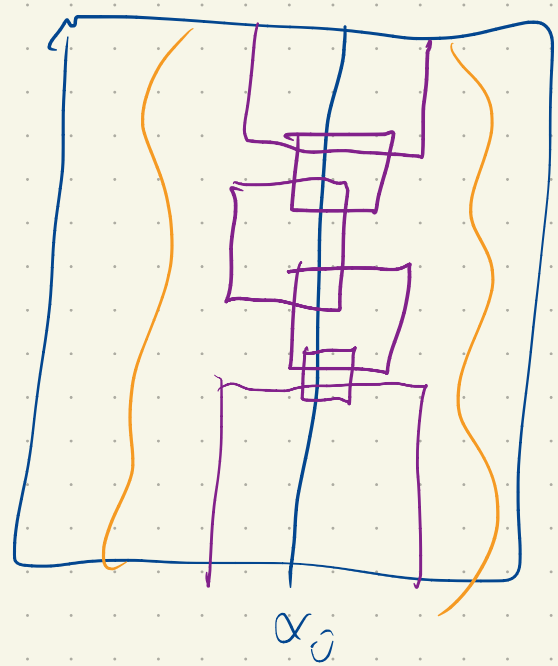
so we can find a finite sub cover $U_{y_1} \times V_{y_1}, \dots, U_{y_n} \times V_{y_n}$.

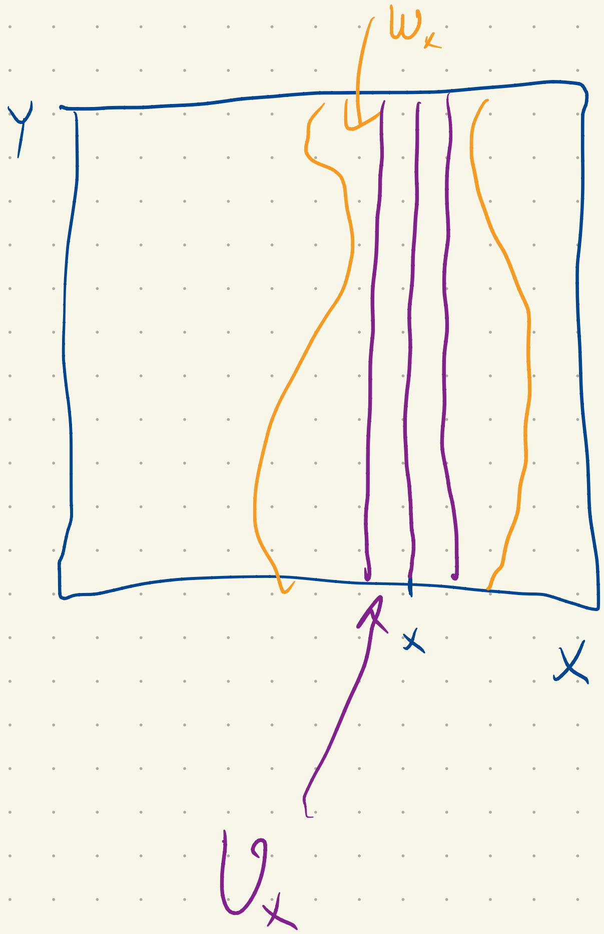
Let $U = \bigcap_{j=1}^{n} U_{y_j}$. Then $U$ is open in $X$ and contains $x_0$.

Suppose $(x,y) \in U \times Y$. Pick $j$ with $(x_0, y) \in U_{y_j} \times V_{y_j}$.

Then $(x,y) \in U \times V_{y_j} \subseteq U_{y_j} \times V_{y_j} \subseteq W$.

So $U \times Y \subseteq W$.






$\{U_\alpha\}$

$\{x\}\times Y$ is covered by finitely many $U_\alpha$'s.

$\pi_X^{-1}(\{x\}) \subseteq \bigcup_{\alpha\in A_x} U_\alpha$ where $A_x$ is finite.

$W_x = \bigcup_{\alpha\in A_x} U_\alpha$ is open and contains $\{x\}\times Y$.

Find $U_x$ with $U_x \times Y \subseteq W_x$

The open sets $U_x$ cover $X$. So we can extract a finite subcover. $U_{x_1},\dots,U_{x_n}$.

$X \subseteq \bigcup U_{x_j}$ and $X \times Y \subseteq \bigcup \pi_X^{-1}(U_{x_j})$

Each $\pi_X^{-1}(U_{x_j})$ is contained in $W_{x_j}$ and hence covered by finitely many sets $U_\alpha$. But there are finitely many tubes $\pi_X^{-1}(U_{x_j})$.

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How about arbitrary products of compact sets?

This is false in the box topology.

$\underbrace{\{0,1\}}_{\text{discrete!}}{}^{\omega}$ is discrete with the box topology.

The singletons are an open cover w/ no finite subcover.

For the product topology this is true! (Tychonoff's Thm).