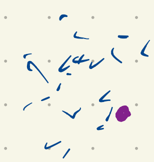
Prop: If X is limit point compact and 1^{st} countable and Hausdorff then it is sequentially compact.

Pf: Let $\{x_n\}$ be a sequence in X. If the sequence takes on only finitely many values then we can extract a constant subsequence. Hence we assume the sequence has infinitely many values and these have a limit point x_0.

n_k

If x is an element of the sequence we reduce to a subsequence where none of the terms are x but still we take on infinitely many values.



Let $\{B_j\}$ be a countable nested neighborhood base at x.

Pick n_1 such that $x_{n_1} \in B_1$. This is possible since x is a limit point of the sequence. Now suppose we have selected $n_1 < n_2 < \cdots < n_k$ such that $x_{n_j} \in B_j$.

Notice that $B_{k+1} \setminus \{x_1, x_2, \dots, x_{n_k}\}$ is open (Hausdorff!)

and contains x. Hence it contains a term of the sequence

$x_{n_{k+1}}$ with $n_{k+1} > n_k$. Inductively we have obtained a

subsequence with $x_{n_j} \in B_j$ for all j.

$\mathcal{U}(x)$

$\uparrow$

all opensets containg x.

Now let U be an open set containing x.

There exists J with $x \in B_J \subseteq U$.

If $j \geq J$ then $x_{n_j} \in B_j \subseteq B_J \subseteq U$.

Hence $x_{n_j} \to x$.

$\square$

Prop: A sequentially compact 2^{nd} countable space X is compact.

Pf: Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X.

Since X is 2^{nd} countable it is Lindelof and we can extract a countable subcover, call it $\{U_j\}$

Suppose to the contrary that there is no finite subcover.

For each j we can then pick $x_j \in X$, $x_j \notin U_1 \cup \cdots \cup U_j$.

The space is sequentially compact, so we can find a convergent subsequence x_{j_k} converging to some x.

Pick J such that $x \in U_J$. Then there exists K so that if $k \geq K$, $x_{j_k} \in U_J$. But $x_{j_k} \notin U_J$ if $j_k \geq J$. So if $k \geq K$ and $j_k \geq J$ (which we can assume by taking k large enough) $x_{j_k} \in U_J$ and $x_{j_k} \notin U_J$, a $\Rightarrow\Leftarrow$

$\mathbb{R}P^2$ is metrizable but no particular metric seems important.

For metric spaces the seq compact $\Rightarrow$ compact result boils down to a question of separability. (i.e whether the space has a countable dense subset.)

Suppose X is a separable metric space. Then it is 2^{nd} countable. So if it is sequentially compact then it is compact.

We'll show shortly that a sequentially compact metric space is in fact separable!

Here, by the previous argument, a sequentially compact metric space is compact. Moreover, if a metric space is not separable then it is not sequentially compact. so it is not compact.

compact $\Rightarrow$ l.p.c $\Rightarrow$ seq. compact

Key play: in a sequentially compact metric space, for all $\varepsilon > 0$ there is a covering by finitely many $\underbrace{\varepsilon\text{-balls}}$

ball of radius ε.

To get separability: For each $\varepsilon = \frac{1}{n}$ find finitely many $x_{n,j}$ such that

$$\bigcup_j B_{\frac{1}{n}}(x_{n,j}) = X.$$

Countably many $x_{n,j}$'s and they are dense.

Lemma: Let X be a sequentially compact metric space. For all $\varepsilon > 0$ there exists a cover of X by finitely many ε-balls.

Pf: Let $\varepsilon > 0$ and suppose to the contrary that no such cover exists. Pick $x_1 \in X$. Since $B_\varepsilon(x_1)$ is not a cover we can find $x_2 \notin B_\varepsilon(x_1)$. Since $\{B_\varepsilon(x_j)\}_{j=1}^{2}$ is not a cover we can find x_3 such that $d(x_3, x_1) \geq \varepsilon$ and $d(x_2, x_1) \geq \varepsilon$. Continuing inductively we can find a sequence $\{x_j\}$ with $d(x_j, x_\ell) \geq \varepsilon$

if $j \neq \ell$. The space is sequentially compact and

we have a convergent subsequence $x_{j_k} \to x$.

Hence we can find K so that if $k > K$,

$d(x_{j_k}, x) < \frac{\varepsilon}{10}$. But then if $k_1, k_2 \geq K$,

$d(x_{j_{k_1}}, x_{j_{k_2}}) \leq \frac{2\varepsilon}{10} = \frac{\varepsilon}{5}$ by the

Δ ineq. But $d(x_{j_{k_1}}, x_{j_{k_2}}) \geq \varepsilon$ if $k_1 \neq k_2$, a

contradiction.