Nets (generalized sequences)

Sequence into X $\qquad \mathbb{N} \to X$

$x(i) \to x_i$

Def: A *directed set* is a set A together with a relation $\leq$ satisfying $\qquad \beta \leq \gamma \Leftrightarrow \gamma \geq \beta$

1) $\alpha \leq \alpha$ for all $\alpha \in A$ (reflexive)

2) $\alpha \leq \beta$ and $\beta \leq \gamma \Rightarrow \alpha \leq \gamma$ for all such $\alpha, \beta, \gamma \in A$ (transitive)

3) If $\alpha, \beta \in A$ there exists $\gamma \in A$ with $\alpha \leq \gamma$, $\beta \leq \gamma$.

1) + 2) almost makes a partial order

$$\alpha \leq \beta \ + \ \beta \leq \alpha \Rightarrow \alpha = \beta$$

Given $\alpha, \beta \in A$
$\alpha \leq \beta$ and $\beta \leq \alpha$
might not hold.

1) $\mathbb{N}, \leq$

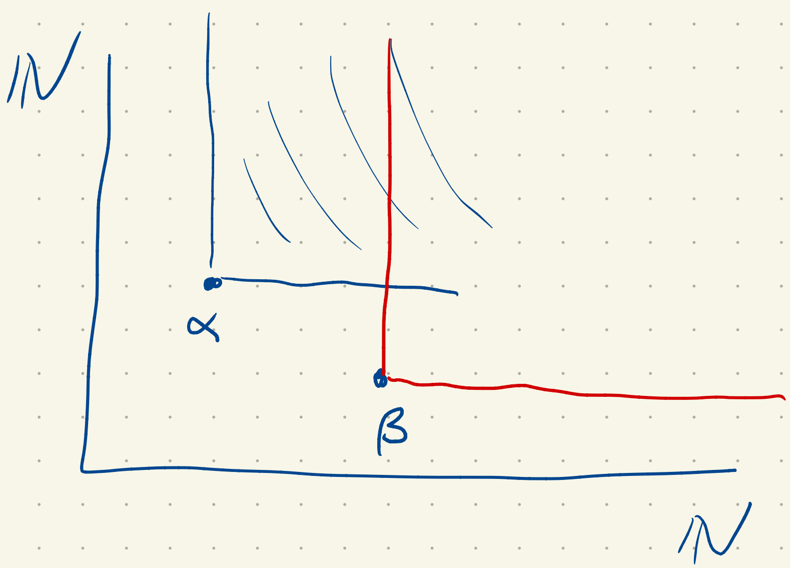
n_1, n_2 $\quad$ $n_1 + n_2$ beats n_1, n_2

$\max(n_1, n_2)$

2) $\mathbb{N} \times \mathbb{N}$ $\quad$ $(a,b) \leq (c,d)$ if $\quad$ $a \leq c$, $b \leq d$

$(1,6) \leq (5, 19)$

$(1,6)$ vs $(2,4)$ $\leftarrow$ not comparable.



3) A is a directed set

$A \times A$ $\quad (a_1, b_1) \le (a_2, b_2)$ if $a_1 \le a_2$
$b_1 \le b_2$

$\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$ $\quad \gamma_1 \ge a_1, a_2$
$\gamma_2 \ge b_1, b_2$

$\uparrow$

$\gamma = (\gamma_1, \gamma_2)$

4) X is a top space. Let $x \in X$.

$A = \mathcal{V}(x)$ open sets contains x

$U \ge V$ if $U \subseteq V$ ($\mathcal{V}(x)$ is ordered by reverse inclusion).

U, V $\quad W = U \cap V$ $\quad W \subseteq U \Rightarrow W \ge U$ and similarly for V.

Def: Let X be a set. A <u>net</u> in X is a function from a directed set A into X.

Remark: Sequences in X are nets in X.

Notation: $x(\alpha) \quad \alpha \in A$

$$\updownarrow$$

x_α

$$\langle x_\alpha \rangle_{\alpha \in A} \qquad \{x_n\}_{n=1}^{\infty}, \ \{x_n\}_{n \in \mathbb{N}}$$

Many topological properties can be characterized in terms of nets.

In metric spaces we can characterize the closure $V \subseteq X$ with sequences. $x \in \overline{V} \iff$ there is a seq. in V converging to x

Prop: Let X be a top. space and let $V \subseteq X$.

Then $x \in \overline{V}$ if and only if there is a net in V converging to x.

Def: Let A be a directed set and let $\alpha_0 \in A$.

Then the tail of α_0 in A, $T(\alpha_0)$, is

$$\{ \alpha \in A : \alpha \geq \alpha_0 \}.$$

$A = \mathbb{N}$ $\qquad$ $T(5) = \{ n \in \mathbb{N} : n \geq 5)$

Exercise: $T(\alpha_0)$ is again a directed set with the same ordering.

Def: Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X.

A tail of the net is a net of the form

$$\langle x_\alpha \rangle_{\alpha \in T(\alpha_0)}$$ for some $\alpha_0 \in A$.

Def: Let X be a top space, $x \in X$, and

$\langle x_\alpha \rangle_{\alpha \in A}$ a net in X. We say $x_\alpha \to x$

($\langle x_\alpha \rangle_{\alpha \in A}$ <u>converges</u> to x) if for every

open set U containing x, U contains a tail

of the net.

Remark: Convergence is equivalent to for all open sets U contains x, there exists α_0 such that if $\alpha \geq \alpha_0$ then $x_\alpha \in U$.

Lemma: Suppose $\langle x_\alpha \rangle$ is a net in $V \subseteq X$ converges to $x \in X$. Then $x \in \overline{V}$.

Pf: Let U be an open set contains x. Since $x_\alpha \to x$, U contains a tail $\langle x_\alpha \rangle_{\alpha \in T(\alpha_0)}$ for some $\alpha_0 \in A$ and in particular contains $x_{\alpha_0} \in V$.

Converse:

Lemma: Let X be a top space, $V \subseteq X$, and $x \in \overline{V}$.

Then there exists a net in V converging to x.

Pf: Let $A = \mathcal{U}(x)$ ordered by reverse inclusion.

For each $U \in A$, since $x \in \overline{V}$, we can pick $x_U \in V \cap U$.

Consider the net $\langle x_U \rangle_{U \in \mathcal{U}(x)}$.

To see that $x_U \to x$ consider an open set W

containing x. If $U \geq W$ (i.e. $U \subseteq W$) then

$x_U \in U \subseteq W$. So W contains the tail $\langle x_U \rangle_{U \in T(W)}$.

$\left(\langle x_U \rangle_{U \geq W} \right)$

Prop: $x \in \overline{V} \iff$ there is a net in V converges to x.

Next up:

$f : X \to Y$ is continuous iff

whenever $\langle x_\alpha \rangle_{\alpha \in A}$ is a net in X converges to some x then

$$\langle f(x_\alpha) \rangle_{\alpha \in A} \text{ converges to } f(x).$$

"f takes convergent nets to convergent nets"

Lemma: If $f : X \to Y$ is continuous then it takes convergent nets to convergent nets

Pf: Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X converges to some x.

We need to show $f(x_\alpha) \to f(x)$.

Let W be an open set containing $f(x)$.

Then $f^{-1}(W)$ is an open set containing x

and hence contains a tail $\langle x_\alpha \rangle_{\alpha \in T(\alpha_0)}$ for some α_0.

But then if $\alpha \geq \alpha_0$, $x_\alpha \in f^{-1}(W)$, so

$f(x_\alpha) \in W$. Hence W contains $\langle f(x_\alpha) \rangle_{\alpha \in T(\alpha_0)}$.

Converse. Lemma: Suppose $f: X \to Y$ takes convergent nets to conv. nets. Then f is continuous.

Pf: Let $V \subseteq Y$ be closed. We need to show that $f^{-1}(V)$ is closed in X.

Let $x \in \overline{f^{-1}(V)}$. There exists a net $\langle x_\alpha \rangle_{\alpha \in A}$ in $f^{-1}(V)$ converging to x.

Since f takes convergent nets to conv. nets,

$$f(x_\alpha) \to f(x).$$

But $\langle f(x_\alpha) \rangle_{\alpha \in A}$ is a net in the closed set V.

Hence $f(x) \in \overline{V} = V$.

So $x \in f^{-1}(V)$, and therefore $f^{-1}(V)$ is closed.