Inverse Methods for Reconstructing Basal Boundary Data

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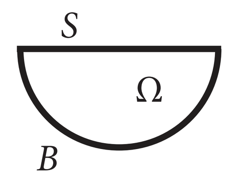
Ill-Posed Boundary Value Problem



We wish to determine conditions at the base B of a glacier from measurements at the surface S. We consider the simplified case of flow through a cross section Ω under the hypotheses that the in-plane velocity components and all out-of-plane gradients are zero. Assuming Glen's flow law (Glen, 1955) for the material stress-strain relationship, the equation for momentum balance in the out of plane direction can be be written in terms of the out of plane velocity component u:

$$\nabla \cdot \left(|\nabla u|^{\frac{1-n}{n}} \nabla u \right) = (2A)^{1/n} \rho g \sin \alpha \tag{1}$$

where ρ is the density of ice, g the acceleration due to gravity, and α the out-of-plane surface slope, and where the flow rate parameter A and the exponent n are empirical constants (we use $n = 3$ in our study). We assume that shear stresses vanish at the surface:

$$\left[|\nabla u|^{\frac{1-n}{n}} \frac{\partial}{\partial \nu} u \right]_S = \tau_S = 0. \tag{2}$$

We also assume that velocities are known on S:

$$u|_S = u_S. \tag{3}$$

Problem [-ND]: Find a solution of (1) satisfying (2) and (3).

Ill-Posedness

Boundary conditions (2) and (3) uniquely determine a solution of (1), but problem [-ND] is ill-posed. Given a smooth surface velocity field u_S, there need not be a solution (although there will be a solution for an infinitesimally perturbed surface velocity field). More significantly, tiny errors in u_S lead to uncontrollably large errors in the corresponding solution of (1). We must use regularization methods to determine approximate solutions, with the quality of the approximation depending nonlinearly on an estimate for our uncertainty in u_S (and in our model (1)).

Well-Posed Auxilliary Problems

Well-posed boundary value problems for (1) require a boundary condition at the base B. Our regularization techniques work with three such conditions:

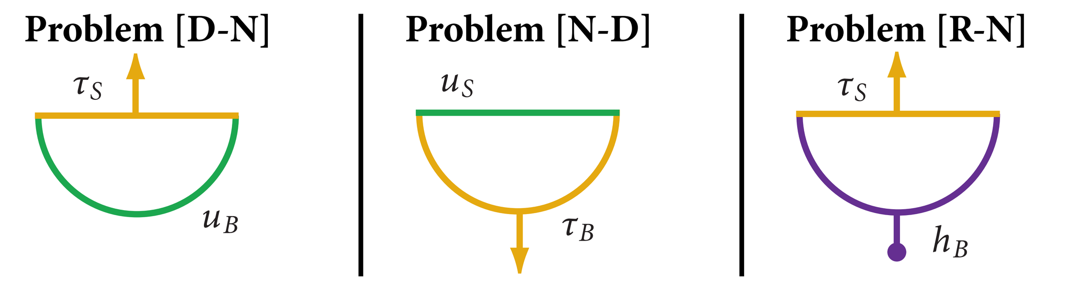
Dirichlet (4)	Neumann (5)	Robin (6)
$u\|_B = u_B$	$\left[\|\nabla u\|^{\frac{1-n}{n}} \frac{\partial}{\partial \nu} u \right]_B = \tau_B$	$\left[\|\nabla u\|^{\frac{1-n}{n}} \frac{\partial}{\partial \nu} u + \gamma u \right]_B = h_B.$

These conditions are used in the following well-posed boundary-value problems.

Problem [D-N]: Find a solution of (1) satisfying (4) on B and (2) on S.

Problem [N-D]: Find a solution of (1) satisfying (5) on B and (3) on S.

Problem [R-N]: Find a solution of (1) satisfying (6) on B and (2) on S.

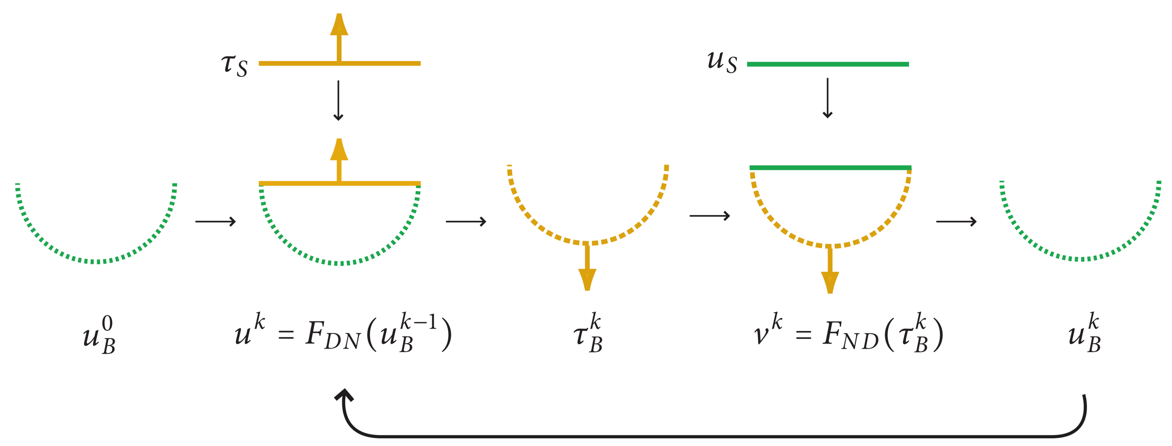


Regularization Methods

We have implemented three regularization methods using or adapting known algorithms for the linear ($n = 1$) version of Problem [-ND] in the inner loop of an iterative procedure to find approximate solutions of the nonlinear problem.

Kozlov-Maz'ya Iteration

Kozlov and Maz'ya (1990) introduced a regularization technique for linear Problem [-ND] that consists of alternately solving Problems [D-N] and [N-D]. The algorithm starts with an initial guess u_B^0 for the basal velocities and continues until the solution of [D-N] matches the surface data u_S to within a specified tolerance that reflects uncertainty in the data and the model.

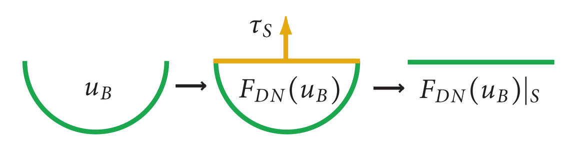


We have found a significant acceleration of the algorithm in the linear case. It forms the core of our fastest regularization method, which was presented in Maxwell and others (2008).

Constrained Dirichlet Problem

The ill-posed Problem [-ND] can be cast in terms of Problem [D-N]: find a basal velocity field u_B such that

$$F_{DN}(u_B)|_S = u_S \tag{7}$$



where F_{DN} is the operator indicated schematically above.

Payne (1975) and Han (1982) studied regularizations of linear Problem [-ND] based on finding the "smoothest" basal data u_B that **approximately** solves (7) (with the error tolerance reflecting confidence in the data and model). Numerically this leads to a constrained quadratic programming problem. We have implemented a variation of this approach, taking advantage of the quadratic programming to enforce an additional constraint: $u_B \geq 0$.

Constrained Robin Problem

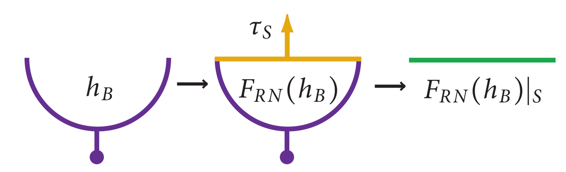
Lin and Fang (2005) used a technique similar to the previous one to solve linear Problem [-ND] when the basal boundary condition is assumed to have had the form

$$[\partial_\nu u + \gamma u]_B = 0$$

where $\gamma \geq 0$ is an unknown Robin coefficient function.

Using an artificial Robin coefficient $\tilde{\gamma}$ that is an upper bound for the unknown Robin coefficient, they seek the "smoothest" solution h_B that approximately solves

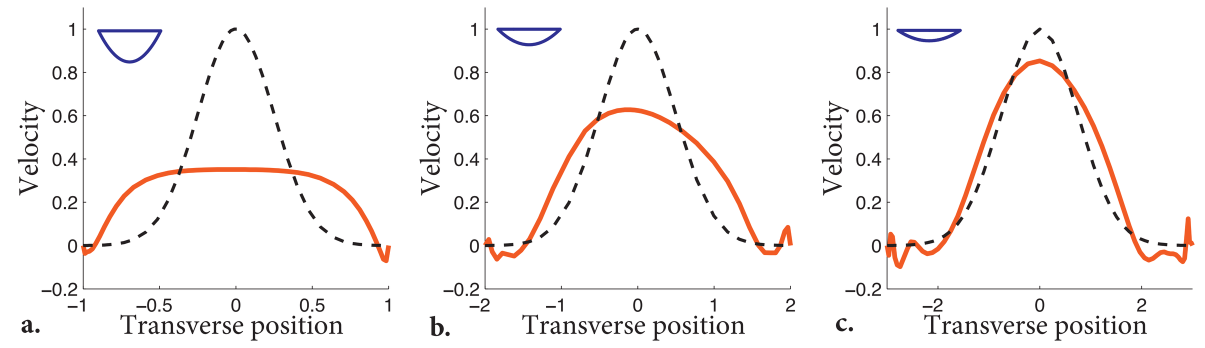
$$F_{RN}(h_B)|_S = u_S. \tag{8}$$



Once a suitable h_B has been determined, they recover $\gamma = \tilde{\gamma} - [h_B / F_{RN}(h_B)]$. The problem of finding h_B is a quadratic programming problem with a constraint on the recovered Robin coefficient: $\gamma \geq 0$. We have implemented this approach with and additional constraint on the basal velocities: $u|_B \geq 0$.

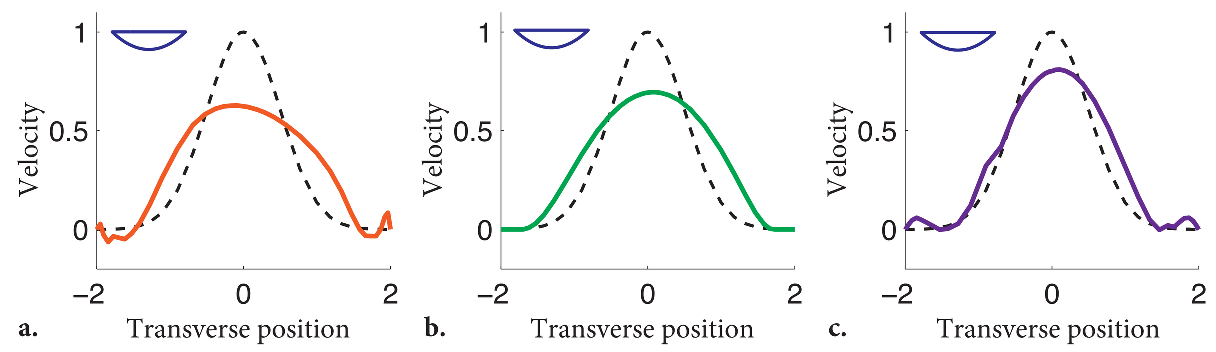
Reconstruction of Synthetic Data

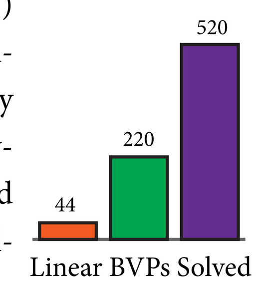
Dependence on Domain Geometry



Reconstruction of basal velocities in domains with parabolic cross sections using Kozlov-Maz'ya iteration. Cross sections have depth equal to 1 and widths W with (a.) $W = 1$, (b.) $W = 2$, and (c.) $W = 3$. Synthetic data generated using a Dirichlet condition at the base specifying a sliding region at the center (dashed line). Noise (2% of peak velocity for a frozen base) was added prior to reconstruction.

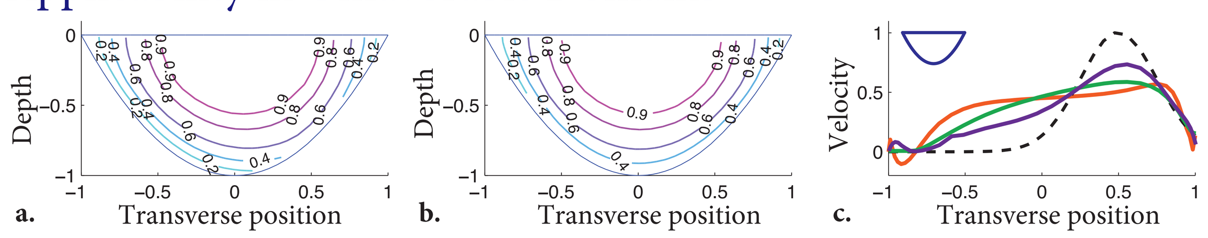
Comparison of Methods





Reconstruction of basal velocities in a parabolic domain using (a.) Kozlov-Maz'ya, (b.) constrained Dirichlet, and (c.) constrained Robin algorithms. All algorithms find solutions that are consistent with the noisy data. Constrained Dirichlet detects the near frozen regions best. Kozlov-Maz'ya is easily the most efficient of the algorithms (fewer linear forward problems solved, and no quadratic programming), but suffers from oscillations near the edges.

Applicability in Unfavorable Domains



Even in domains where resolution at the base is limited, much of the interior solution can be reconstructed. (a.) Contours of synthetic velocities with sliding on the right side of the base. (b.) Reconstruction using Kozlov-Maz'ya iteration. (c.) Reconstruction of velocites at the base: Kozlov-Maz'ya (orange), constrained Dirichlet (green), constrained Robin (purple), true (black dashed).

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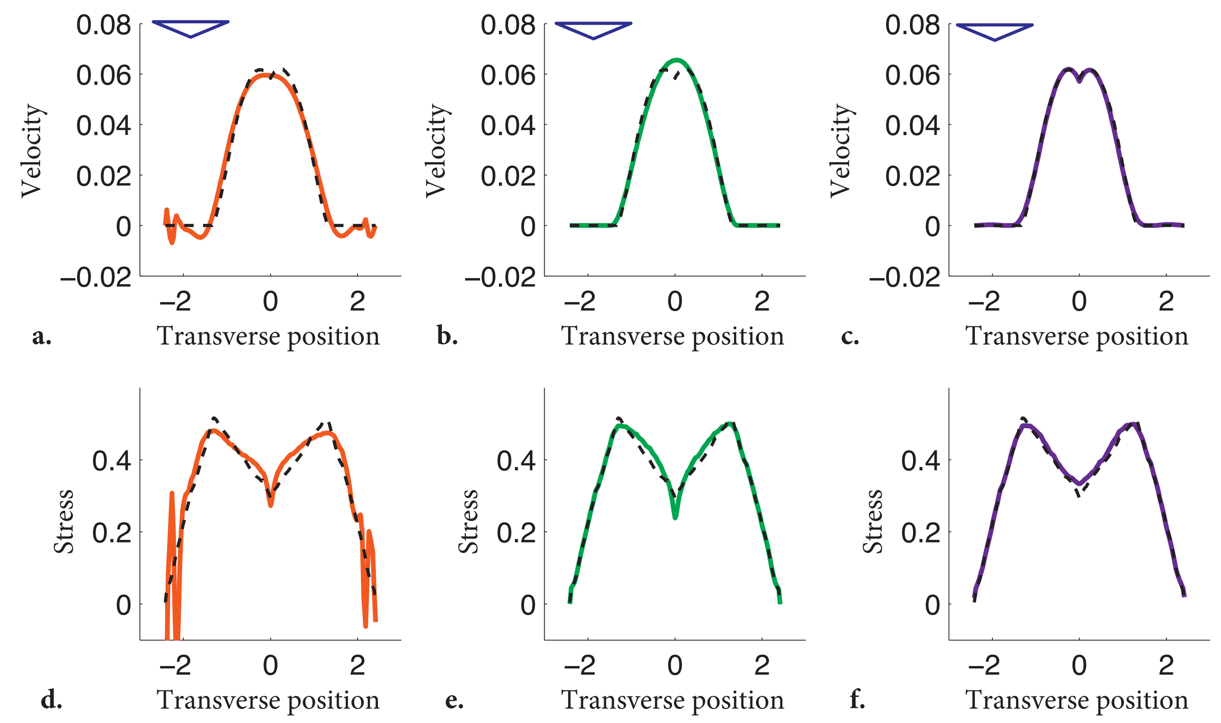
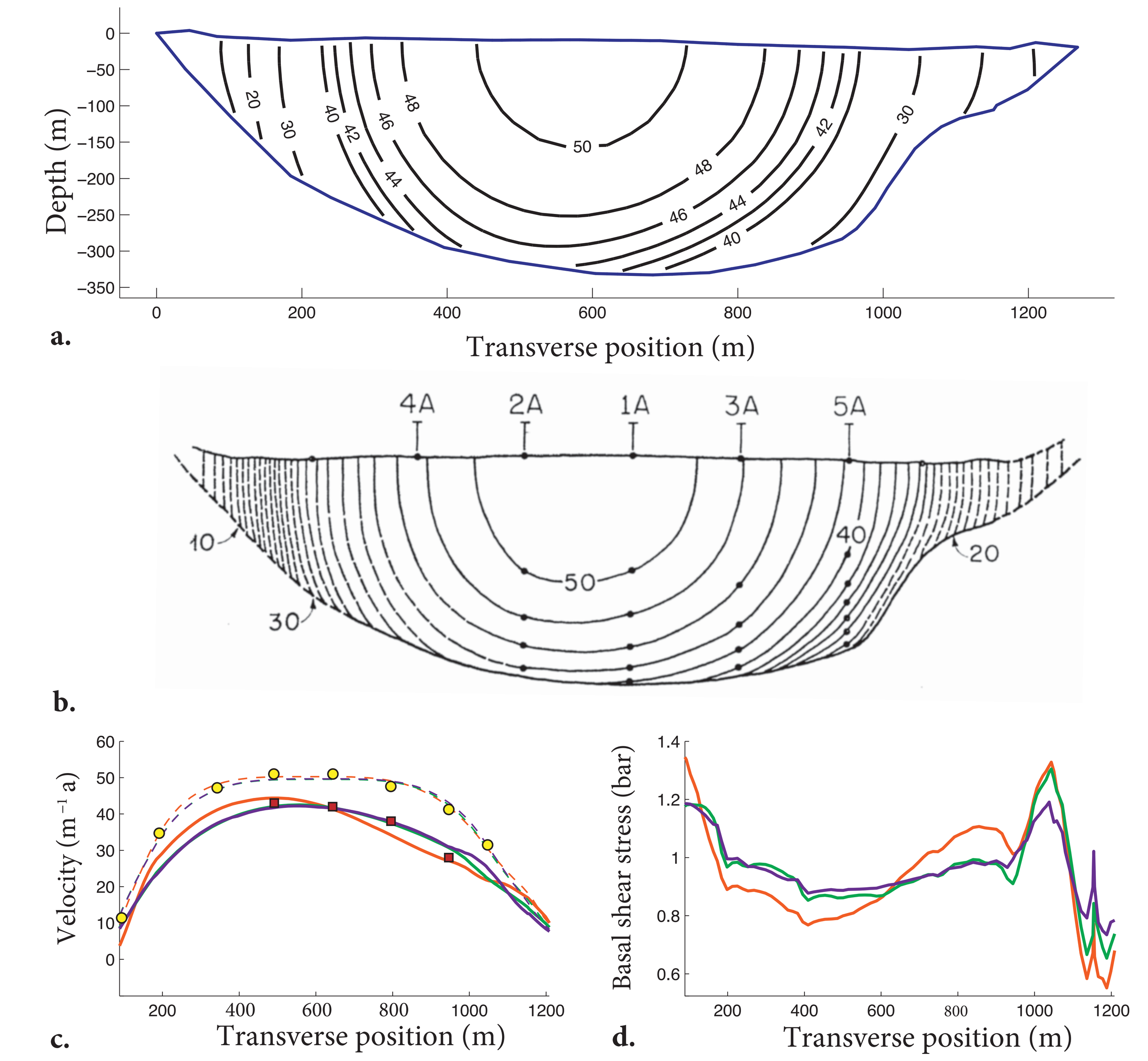
Reconstruction of Synthetic Data

Coulomb Friction



Reconstruction of basal velocities (a.–c.) and stresses (d.–f.) for synthetic data generated using a Coulomb friction condition on B. Ice is frozen to the base unless the basal shear stress reaches a depth dependent yield stress. Domain and yield stresses taken from an example in Schoof (2006). Kozlov-Maz'ya reconstruction (a., d.) suffers from velocity and stress oscillations. Constrained Dirichlet (b., e.) and constrained Robin (c., f.) reconstructions accurately detect frozen areas as well as yield stress where ice is sliding. Superior constrained Robin construction reflects a bias towards a simpler Robin coefficient over a simpler velocity distribution.

Reconstruction for Athabasca Glacier



(a.) Reconstructed velocity contour lines (m a^{-1}) using Kozlov-Maz'ya iteration applied to surface measurements from Raymond (1971). (b.) Contour lines derived from measurements (Raymond, 1971). (c.) Measured (circles) and reconstructed (dashed lines) surface velocities, and measurement-derived (squares) and reconstructed basal velocities (solid lines). (d.) Reconstructed basal shear stress. Kozlov-Maz'ya (orange), constrained Dirichlet (green), constrained Robin (purple). Reconstruction depends highly on the value of A in (1). Athabasca data are not consistent with transverse flow with Glen parameters $n = 3$ and $A = 0.1 \text{bar}^{-3}\text{a}^{-1}$. We used $A = 0.063 \text{bar}^{-3}\text{a}^{-1}$.